Contents 3

Equations, Inequalities & Partial Fractions

Learning outcomes

In this Workbook you will learn about solving single equations, mainly linear and quadratic, but also cubic and higher degree, and also simultaneous linear equations. Such equations often arise as part of a more complicated problem. In order to gain confidence in mathematics you will need to be thoroughly familiar with these basis topics.
You will also study how to manipulate inequalities. You will also be introduced to partial fractions which will enable you to re-express an algebraic fraction in terms of simpler fractions. This will prove to be extremely useful in later studies on integration.

Solving Linear Equations

3.1

Introduction

Many problems in engineering reduce to the solution of an equation or a set of equations. An equation is a type of mathematical expression which contains one or more unknown quantities which you will be required to find. In this Section we consider a particular type of equation which contains a single unknown quantity, and is known as a linear equation. Later Sections will describe techniques for solving other types of equations.

Prerequisites

Before starting this Section you should ...

- be able to add, subtract, multiply and divide fractions
- be able to transpose formulae

Learning Outcomes

On completion you should be able to ...

- recognise and solve a linear equation

1. Linear equations

Key Point 1

A **linear equation** is an equation of the form

$$ax + b = 0 \qquad a \neq 0$$

where a and b are known numbers and x represents an unknown quantity to be found.

In the equation $ax + b = 0$, the number a is called the **coefficient of** x, and the number b is called the **constant term**.

The following are examples of linear equations

$$3x + 4 = 0, \quad -2x + 3 = 0, \quad -\frac{1}{2}x - 3 = 0$$

Note that the unknown, x, appears only to the first power, that is as x, and not as x^2, $\sqrt{x}$, $x^{1/2}$ etc. Linear equations often appear in a non-standard form, and also different letters are sometimes used for the unknown quantity. For example

$$2x = x + 1 \qquad 3t - 7 = 17, \qquad 13 = 3z + 1, \qquad 1 - \frac{1}{2}y = 3 \qquad 2\alpha - 1.5 = 0$$

are all examples of linear equations. Where necessary the equations can be rearranged and written in the form $ax + b = 0$. We will explain how to do this later in this Section.

Task

Which of the following are linear equations and which are not linear?

(a) $3x + 7 = 0$, (b) $-3t + 17 = 0$, (c) $3x^2 + 7 = 0$, (d) $5p = 0$

The equations which can be written in the form $ax + b = 0$ are linear.

Your solution

(a) (b) (c) (d)

Answer

(a) linear in x (b) linear in t (c) non-linear - quadratic in x (d) linear in p, constant is zero

To solve a linear equation means to find the value of x that can be substituted into the equation so that the left-hand side equals the right-hand side. Any such value obtained is known as a **solution** or **root** of the equation and the value of x is said to **satisfy** the equation.

Example 1

Consider the linear equation $3x - 2 = 10$.

(a) Check that $x = 4$ is a solution.

(b) Check that $x = 2$ is *not* a solution.

Solution

(a) To check that $x = 4$ is a solution we substitute the value for x and see if both sides of the equation are equal. Evaluating the left-hand side we find $3(4) - 2$ which equals 10, the same as the right-hand side. So, $x = 4$ is a solution. We say that $x = 4$ satisfies the equation.

(b) Substituting $x = 2$ into the left-hand side we find $3(2) - 2$ which equals 4. Clearly the left-hand side is not equal to 10 and so $x = 2$ is not a solution. The number $x = 2$ does not satisfy the equation.

Task

Test which of the given values are solutions of the equation

$$18 - 4x = 26$$

(a) $x = 2$, (b) $x = -2$, (c) $x = 8$

(a) Substituting $x = 2$, the left-hand side equals

Your solution

Answer

$18 - 4 \times 2 = 10$. But $10 \neq 26$ so $x = 2$ is not a solution.

(b) Substituting $x = -2$, the left-hand side equals:

Your solution

Answer

$18 - 4(-2) = 26$. This is the same as the right-hand side, so $x = -2$ is a solution.

(c) Substituting $x = 8$, the left-hand side equals:

Your solution

Answer

$18 - 4(8) = -14$. But $-14 \neq 26$ and so $x = 8$ is not a solution.

Exercises

1. (a) Write down the general form of a linear equation.
   (b) Explain what is meant by the root or solution of a linear equation.

In questions 2-8 verify that the given value is a solution of the given equation.

2. $3z - 7 = -28, \quad z = -7$
3. $8x - 3 = -11, \quad x = -1$
4. $2s + 3 = 4, \quad s = \frac{1}{2}$
5. $\frac{1}{3}x + \frac{4}{3} = 2, \quad x = 2$
6. $7t + 7 = 7, \quad t = 0$
7. $11x - 1 = 10, \quad x = 1$
8. $0.01t - 1 = 0, \quad t = 100.$

Answers

1. (a) The general form is $ax + b = 0$ where a and b are known numbers and x represents the unknown quantity.
   (b) A root is a value for the unknown which satisfies the equation.

2. Solving a linear equation

To solve a linear equation we make the unknown quantity the **subject** of the equation. We obtain the unknown quantity on its own on the left-hand side. To do this we may apply the same rules used for transposing formulae given in Workbook 1 Section 1.7. These are given again here.

Key Point 2

Operations which can be used in the process of solving a linear equation

- add the same quantity to both sides
- subtract the same quantity from both sides
- multiply both sides by the same quantity
- divide both sides by the same quantity
- take the reciprocal of both sides (invert)
- take functions of both sides; for example cube both sides.

A useful summary of the rules in Key Point 2 is 'whatever we do to one side of an equation we must also do to the other'.

Example 2

Solve the equation $x + 14 = 5$.

Solution

Note that by subtracting 14 from both sides, we leave x on its own on the left. Thus

$$\begin{aligned} x + 14 - 14 &= 5 - 14 \\ x &= -9 \end{aligned}$$

Hence the solution of the equation is $x = -9$. It is easy to check that this solution is correct by substituting $x = -9$ into the original equation and checking that both sides are indeed the same. You should get into the habit of doing this.

Example 3

Solve the equation $19y = 38$.

Solution

In order to make y the subject of the equation we can divide both sides by 19:

$$\begin{aligned} 19y &= 38 \\ \frac{19y}{19} &= \frac{38}{19} \\ \text{cancelling 19's gives} \qquad y &= \frac{38}{19} \\ \text{so} \qquad y &= 2 \end{aligned}$$

Hence the solution of the equation is $y = 2$.

Example 4

Solve the equation $4x + 12 = 0$.

Solution

Starting from $4x + 12 = 0$ we can subtract 12 from both sides to obtain

$$\begin{aligned} 4x + 12 - 12 &= 0 - 12 \\ \text{so that} \qquad 4x &= -12 \end{aligned}$$

If we now divide both sides by 4 we find

$$\begin{aligned} \frac{4x}{4} &= \frac{-12}{4} \\ \text{cancelling 4's gives} \qquad x &= -3 \end{aligned}$$

So the solution is $x = -3$.

Task

Solve the linear equation $14t - 56 = 0$.

Your solution

Answer

$t = 4$

Example 5

Solve the following equations: (a) $x + 3 = \sqrt{7}$, (b) $x + 3 = -\sqrt{7}$.

Solution

(a) Subtracting 3 from both sides gives $x = \sqrt{7} - 3$.

(b) Subtracting 3 from both sides gives $x = -\sqrt{7} - 3$.

Note that when asked to solve $x + 3 = \pm\sqrt{7}$ we can write the two solutions as $x = -3 \pm \sqrt{7}$. It is usually acceptable to leave the solutions in this form (i.e. with the $\sqrt{7}$ term) rather than calculate decimal approximations. This form is known as the **surd form**.

Example 6
Solve the equation $\frac{2}{3}(t+7) = 5$.

Solution

There are a number of ways in which the solution can be obtained. The idea is to gradually remove unwanted terms on the left-hand side to leave t on its own. By multiplying both sides by $\frac{3}{2}$ we find

$\frac{3}{2} \times \frac{2}{3}(t+7) = \frac{3}{2} \times 5 = \frac{3}{2} \times \frac{5}{1}$ and after simplifying and cancelling, $\quad t + 7 = \frac{15}{2}$

Finally, subtracting 7 from both sides gives

$$t = \frac{15}{2} - 7 = \frac{15}{2} - \frac{14}{2} = \frac{1}{2}$$

So the solution is $t = \frac{1}{2}$.

Example 7
Solve the equation $3(p-2) + 2(p+4) = 5$.

Solution

At first sight this may not appear to be in the form of a linear equation. Some preliminary work is necessary. Removing the brackets and collecting like terms we find the left-hand side yields $5p + 2$ so the equation is $5p + 2 = 5$ so that $p = \frac{3}{5}$.

Task

Solve the equation $2(x-5) = 3 - (x+6)$.

(a) First remove the brackets on both sides:

Your solution

Answer

$2x - 10 = 3 - x - 6$. We may write this as $2x - 10 = -x - 3$.

(b) Rearrange the equation found in (a) so that terms involving x appear only on the left-hand side, and constants on the right. Start by adding 10 to both sides:

Your solution

Answer

$2x = -x + 7$

(c) Now add x to both sides:

Your solution

Answer

$3x = 7$

(d) Finally solve this to find x:

Your solution

$x =$

Answer

$\frac{7}{3}$

Example 8

Solve the equation

$$\frac{6}{1-2x} = \frac{7}{x-2}$$

Solution

This equation appears in an unfamiliar form but it can be rearranged into the standard form of a linear equation. By multiplying both sides by $(1-2x)$ and $(x-2)$ we find

$$(1-2x)(x-2) \times \frac{6}{1-2x} = (1-2x)(x-2) \times \frac{7}{x-2}$$

Considering each side in turn and cancelling common factors:

$$6(x-2) = 7(1-2x)$$

Removing the brackets and rearranging to find x we have

$$6x - 12 = 7 - 14x$$

Further rearrangement gives: $$20x = 19$$

The solution is therefore $x = \dfrac{19}{20}$.

Example 9

Figure 1 shows three branches of an electrical circuit which meet together at x. Point x is known as a **node**. As shown in Figure 1 the current in each of the branches is denoted by I, I_1 and I_2. Kirchhoff's current law states that the current entering any node must equal the current leaving that node. Thus we have the equation $I = I_1 + I_2$

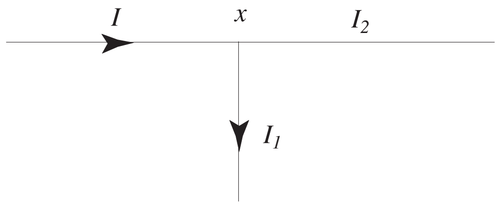

Figure 1

(a) Given $I_2 = 10$ A and $I = 18$ A calculate I_1.

(b) Suppose $I = 36$ A and it is known that current I_2 is five times as great as I_1. Find the branch currents.

Solution

(a) Substituting the given values into the equation we find $18 = I_1 + 10$.

Solving for I_1 we find

$$I_1 = 18 - 10 = 8$$

Thus I_1 equals 8 A.

(b) From Kirchhoff's law, $I = I_1 + I_2$.

We are told that I_2 is five times as great as I_1, and so we can write $I_2 = 5I_1$.

Since $I = 36$ we have

$$36 = I_1 + 5I_1$$

Solving this linear equation $36 = 6I_1$ gives $I_1 = 6$ A.

Finally, since I_2 is five times as great as I_1, we have $I_2 = 5I_1 = 30$ A.

Exercises

In questions 1-24 solve each equation:

1. $7x = 14$
2. $-3x = 6$
3. $\frac{1}{2}x = 7$
4. $3x = \frac{1}{2}$
5. $4t = -2$
6. $2t = 4$
7. $4t = 2$
8. $2t = -4$
9. $\dfrac{x}{6} = 3$
10. $\dfrac{x}{6} = -3$
11. $7x + 2 = 9$
12. $7x + 2 = 23$
13. $-7x + 1 = -6$
14. $-7x + 1 = -13$
15. $\dfrac{17}{3}t = -2$
16. $3 - x = 2x + 8$
17. $x - 3 = 8 + 3x$
18. $\dfrac{x}{4} = 16$
19. $\dfrac{x}{9} = -2$
20. $-\dfrac{13}{2}x = 14$
21. $-2y = -6$
22. $-7y = 11$
23. $-69y = -690$
24. $-8 = -4\gamma$.

In questions 25-47 solve each equation:

25. $3y - 8 = \dfrac{1}{2}y$
26. $7t - 5 = 4t + 7$
27. $3x + 4 = 4x + 3$
28. $4 - 3x = 4x + 3$
29. $3x + 7 = 7x + 2$
30. $3(x + 7) = 7(x + 2)$
31. $2x - 1 = x - 3$
32. $2(x + 4) = 8$
33. $-2(x - 3) = 6$
34. $-2(x - 3) = -6$
35. $-3(3x - 1) = 2$
36. $2 - (2t + 1) = 4(t + 2)$
37. $5(m - 3) = 8$
38. $5m - 3 = 5(m - 3) + 2m$
39. $2(y + 1) = -8$
40. $17(x - 2) + 3(x - 1) = x$
41. $\dfrac{1}{3}(x + 3) = -9$
42. $\dfrac{3}{m} = 4$
43. $\dfrac{5}{m} = \dfrac{2}{m + 1}$
44. $-3x + 3 = 18$
45. $3x + 10 = 31$
46. $x + 4 = \sqrt{8}$
47. $x - 4 = \sqrt{23}$
48. If $y = 2$ find x if $4x + 3y = 9$
49. If $y = -2$ find x if $4x + 5y = 3$
50. If $y = 0$ find x if $-4x + 10y = -8$
51. If $x = -3$ find y if $2x + y = 8$
52. If $y = 10$ find x when $10x + 55y = 530$
53. If $\gamma = 2$ find β if $54 = \gamma - 4\beta$

In questions 54-63 solve each equation:

54. $\dfrac{x - 5}{2} - \dfrac{2x - 1}{3} = 6$
55. $\dfrac{x}{4} + \dfrac{3x}{2} - \dfrac{x}{6} = 1$
56. $\dfrac{x}{2} + \dfrac{4x}{3} = 2x - 7$
57. $\dfrac{5}{3m + 2} = \dfrac{2}{m + 1}$
58. $\dfrac{2}{3x - 2} = \dfrac{5}{x - 1}$
59. $\dfrac{x - 3}{x + 1} = 4$
60. $\dfrac{x + 1}{x - 3} = 4$
61. $\dfrac{y - 3}{y + 3} = \dfrac{2}{3}$
62. $\dfrac{4x + 5}{6} - \dfrac{2x - 1}{3} = x$
63. $\dfrac{3}{2s - 1} + \dfrac{1}{s + 1} = 0$
64. Solve the linear equation $ax + b = 0$ to find x
65. Solve the linear equation $\dfrac{1}{ax + b} = \dfrac{1}{cx + d}$ $(a \neq c)$ to find x

Answers

1. 2	2. -2	3. 14	4. $1/6$	5. $-1/2$	6. 2
7. $1/2$	8. -2	9. 18	10. -18	11. 1	12. 3
13. 1	14. 2	15. $-6/17$	16. $-5/3$	17. $-11/2$	18. 64
19. -18	20. $-28/13$	21. $y = 3$	22. $-11/7$	23. $y = 10$	24. 2
25. $16/5$	26. 4	27. 1	28. 1/7	29. $5/4$	30. 7/4
31. -2	32. 0	33. 0	34. 6	35. $1/9$	36. $-7/6$
37. $23/5$	38. 6	39. -5	40. 37/19	41. -30	42. 3/4
43. $-5/3$	44. -5	45. 7	46. $\sqrt{8} - 4$	47. $\sqrt{23} + 4$	48. 3/4
49. 13/4	50. 2	51. 14	52. -2	53. -13	54. -49
55. 12/19	56. 42	57. 1	58. 8/13	59. $-7/3$	60. 13/3
61. 15	62. 7/6	63. $-2/5$	64. $-b/a$	65. $\dfrac{(d-b)}{(a-c)}$	