The Scalar Product

9.3

Introduction

There are two kinds of multiplication involving vectors. The first is known as the **scalar product** or **dot product**. This is so-called because when the scalar product of two vectors is calculated the result is a scalar. The second product is known as the **vector product**. When this is calculated the result is a vector. The definitions of these products may seem rather strange at first, but they are widely used in applications. In this Section we consider only the scalar product.

Prerequisites

Before starting this Section you should . . .

- know that a vector can be represented as a directed line segment
- know how to express a vector in Cartesian form
- know how to find the modulus of a vector

Learning Outcomes

On completion you should be able to . . .

- calculate, from its definition, the scalar product of two given vectors
- calculate the scalar product of two vectors given in Cartesian form
- use the scalar product to find the angle between two vectors
- use the scalar product to test whether two vectors are perpendicular

1. Definition of the scalar product

Consider the two vectors $\underline{a}$ and $\underline{b}$ shown in Figure 29.

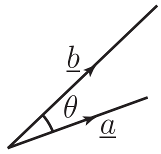

Figure 29: Two vectors subtend an angle θ

Note that the tails of the two vectors coincide and that the angle between the vectors is labelled θ. Their scalar product, denoted by $\underline{a} \cdot \underline{b}$, is defined as the product $|\underline{a}|\,|\underline{b}|\,\cos\theta$. It is very important to use the **dot** in the formula. The dot is the specific symbol for the scalar product, and is the reason why the scalar product is also known as the **dot product**. You should not use a $\times$ sign in this context because this sign is reserved for the vector product which is quite different.

The angle θ is always chosen to lie between 0 and π, and the tails of the two vectors must coincide. Figure 30 shows two incorrect ways of measuring θ.

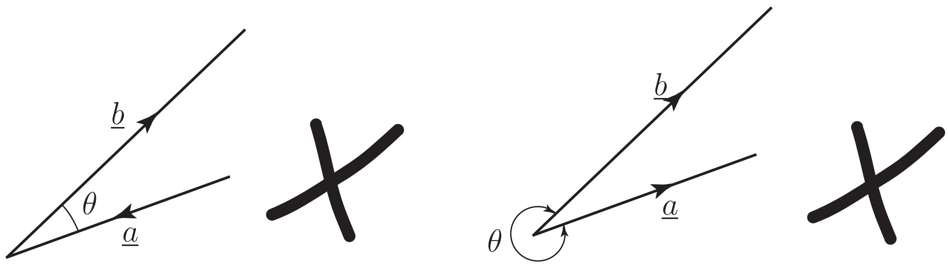

Figure 30: θ should not be measured in these ways

Key Point 9

The scalar product of $\underline{a}$ and $\underline{b}$ is: $\quad \underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta$

We can remember this formula as:

"The modulus of the first vector, multiplied by the modulus of the second vector, multiplied by the cosine of the angle between them."

Clearly $\underline{b} \cdot \underline{a} = |\underline{b}|\,|\underline{a}|\,\cos\theta$ and so

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}.$$

Thus we can evaluate a scalar product in any order: the operation is **commutative**.

Example 8

Vectors $\underline{a}$ and $\underline{b}$ are shown in the Figure 31. Vector $\underline{a}$ has modulus 6 and vector $\underline{b}$ has modulus 7 and the angle between them is $60°$. Calculate $\underline{a}.\underline{b}$.

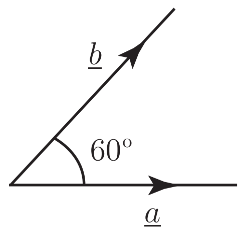

Figure 31

Solution

The angle between the two vectors is $60°$. Hence

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta = (6)(7)\cos 60° = 21$$

The scalar product of $\underline{a}$ and $\underline{b}$ is 21. Note that a scalar product is always a scalar.

Example 9

Find $\underline{i} \cdot \underline{i}$ where $\underline{i}$ is the unit vector in the direction of the positive x axis.

Solution

Because $\underline{i}$ is a unit vector its modulus is 1. Also, the angle between $\underline{i}$ and itself is zero. Therefore

$$\underline{i}.\underline{i} = (1)(1)\cos 0° = 1$$

So the scalar product of $\underline{i}$ with itself equals 1. It is easy to verify that $\underline{j}.\underline{j} = 1$ and $\underline{k}.\underline{k} = 1$.

Example 10

Find $\underline{i} \cdot \underline{j}$ where $\underline{i}$ and $\underline{j}$ are unit vectors in the directions of the x and y axes.

Solution

Because $\underline{i}$ and $\underline{j}$ are unit vectors they each have a modulus of 1. The angle between the two vectors is $90°$. Therefore

$$\underline{i} \cdot \underline{j} = (1)(1)\cos 90° = 0$$

That is $\underline{i} \cdot \underline{j} = 0$.

The following results are easily verified:

Key Point 10

$$\begin{aligned}\underline{i}\cdot\underline{i} &= \underline{j}\cdot\underline{j}=\underline{k}\cdot\underline{k}=1\\ \underline{i}\cdot\underline{j} &= \underline{j}\cdot\underline{i}=0\\ \underline{i}\cdot\underline{k} &= \underline{k}\cdot\underline{i}=0\\ \underline{j}\cdot\underline{k} &= \underline{k}\cdot\underline{j}=0\end{aligned}$$

Generally, whenever any two vectors are perpendicular to each other their scalar product is zero because the angle between the vectors is 90° and $\cos 90^\circ = 0$.

Key Point 11

The scalar product of perpendicular vectors is zero.

2. A formula for finding the scalar product

We can use the results summarized in Key Point 10 to obtain a formula for finding a scalar product when the vectors are given in Cartesian form. We consider vectors in the xy plane. Suppose $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$. Then

$$\begin{aligned}\underline{a}\cdot\underline{b} &= (a_1\underline{i}+a_2\underline{j})\cdot(b_1\underline{i}+b_2\underline{j})\\ &= a_1\underline{i}\cdot(b_1\underline{i}+b_2\underline{j})+a_2\underline{j}\cdot(b_1\underline{i}+b_2\underline{j})\\ &= a_1b_1\underline{i}\cdot\underline{i}+a_1b_2\underline{i}\cdot\underline{j}+a_2b_1\underline{j}\cdot\underline{i}+a_2b_2\underline{j}\cdot\underline{j}\end{aligned}$$

Using the results in Key Point 10 we can simplify this to give the following formula:

Key Point 12

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$ then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2$$

Thus to find the scalar product of two vectors their $\underset{\sim}{i}$ components are multiplied together, their $\underset{\sim}{j}$ components are multiplied together and the results are added.

Example 11

If $\underset{\sim}{a} = 7\underset{\sim}{i} + 8\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$, find the scalar product $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

Solution

We use Key Point 12:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (7\underset{\sim}{i} + 8\underset{\sim}{j}) \cdot (5\underset{\sim}{i} - 2\underset{\sim}{j}) = (7)(5) + (8)(-2) = 35 - 16 = 19$$

The formula readily generalises to vectors in three dimensions as follows:

Key Point 13

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 12

If $\underset{\sim}{a} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 8\underset{\sim}{i} - 9\underset{\sim}{j} + 11\underset{\sim}{k}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

Solution

We use the formula in Key Point 13:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (5)(8) + (3)(-9) + (-2)(11) = 40 - 27 - 22 = -9$$

Note again that the result is a scalar: there are no $\underset{\sim}{i}$'s, $\underset{\sim}{j}$'s, or $\underset{\sim}{k}$'s in the answer.

Task

If $\underline{p} = 4\underline{i} - 3\underline{j} + 7\underline{k}$ and $\underline{q} = 6\underline{i} - \underline{j} + 2\underline{k}$, find $\underline{p} \cdot \underline{q}$.

Use Key Point 13:

Your solution

Answer

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Task

If $\underline{r} = 3\underline{i} + 2\underline{j} + 9\underline{k}$ find $\underline{r} \cdot \underline{r}$. Show that this is the same as $|\underline{r}|^2$.

Your solution

Answer

$\underline{r} \cdot \underline{r} = (3\underline{i} + 2\underline{j} + 9\underline{k}) \cdot (3\underline{i} + 2\underline{j} + 9\underline{k}) = 3\underline{i} \cdot 3\underline{i} + 3\underline{i} \cdot 2\underline{j} + \cdots = 9 + 0 + \cdots = 94.$

$|\underline{r}| = \sqrt{9 + 4 + 81} = \sqrt{94}$, hence $|\underline{r}|^2 = \underline{r} \cdot \underline{r}$.

The above result is generally true:

Key Point 14

For any vector $\underline{r}$, $\quad |\underline{r}|^2 = \underline{r} \cdot \underline{r}$

3. Resolving one vector along another

The scalar product can be used to find the component of a vector in the direction of another vector. Consider Figure 32 which shows two arbitrary vectors $\underline{a}$ and $\underline{n}$. Let $\hat{\underline{n}}$ be a **unit vector** in the direction of $\underline{n}$.

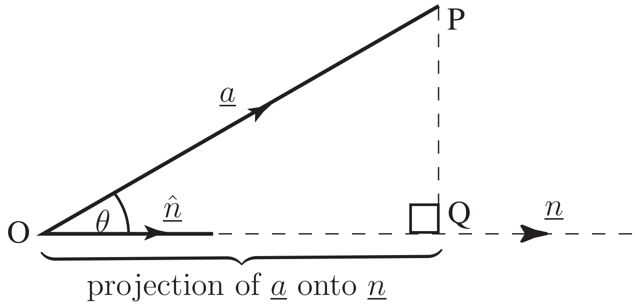

Figure 32

Study the figure carefully and note that a perpendicular has been drawn from P to meet $\underline{n}$ at Q. The distance OQ is called the **projection** of $\underline{a}$ onto $\underline{n}$. Simple trigonometry tells us that the length of the projection is $|\underline{a}|\cos\theta$. Now by taking the scalar product of $\underline{a}$ with the unit vector $\hat{\underline{n}}$ we find

$$\underline{a}\cdot\hat{\underline{n}} = |\underline{a}|\,|\hat{\underline{n}}|\,\cos\theta = |\underline{a}|\cos\theta \qquad \text{(since } |\hat{\underline{n}}| = 1)$$

We conclude that

Key Point 15

Resolving One Vector Along Another

$\underline{a}\cdot\hat{\underline{n}}$ is the component of $\underline{a}$ in the direction of $\underline{n}$

Example 13

Figure 33 shows a plane containing the point A which has position vector $\underline{a}$. The vector $\hat{\underline{n}}$ is a unit vector perpendicular to the plane (such a vector is called a **normal** vector). Find an expression for the perpendicular distance, ℓ, of the plane from the origin.

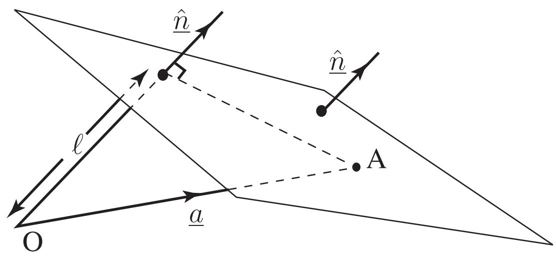

Figure 33

Solution

From the diagram we note that the perpendicular distance ℓ of the plane from the origin is the projection of $\underline{a}$ onto $\hat{\underline{n}}$ and, using Key Point 15, is thus $\underline{a} \cdot \hat{\underline{n}}$.

4. Using the scalar product to find the angle between vectors

We have two distinct ways of calculating the scalar product of two vectors. From Key Point 9 $\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta$ whilst from Key Point 13 $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$. Both methods of calculating the scalar product are entirely equivalent and will always give the same value for the scalar product. We can exploit this correspondence to find the angle between two vectors. The following example illustrates the procedure to be followed.

Example 14

Find the angle between the vectors $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$.

Solution

The scalar product of these two vectors has already been found in Example 12 to be -9. The modulus of $\underline{a}$ is $\sqrt{5^2 + 3^2 + (-2)^2} = \sqrt{38}$. The modulus of $\underline{b}$ is $\sqrt{8^2 + (-9)^2 + 11^2} = \sqrt{266}$. Substituting these values for $\underline{a} \cdot \underline{b}$, $|\underline{a}|$ and $\underline{b}$ into the formula for the scalar product we find

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}|\,|\underline{b}|\,\cos\theta \\ -9 &= \sqrt{38}\,\sqrt{266}\cos\theta \end{aligned}$$

from which

$$\cos\theta = \frac{-9}{\sqrt{38}\sqrt{266}} = -0.0895$$

so that $\theta = \cos^{-1}(-0.0895) = 95.14^\circ$

In general, the angle between two vectors can be found from the following formula:

Key Point 16

The angle θ between vectors $\underline{a}$, $\underline{b}$ is such that:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|}$$

Exercises

1. If $\underline{a} = 2\underline{i} - 5\underline{j}$ and $\underline{b} = 3\underline{i} + 2\underline{j}$ find $\underline{a} \cdot \underline{b}$ and verify that $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.
2. Find the angle between $\underline{p} = 3\underline{i} - \underline{j}$ and $\underline{q} = -4\underline{i} + 6\underline{j}$.
3. Use the definition of the scalar product to show that if two vectors are perpendicular, their scalar product is zero.
4. If $\underline{a}$ and $\underline{b}$ are perpendicular, simplify $(\underline{a} - 2\underline{b}) \cdot (3\underline{a} + 5\underline{b})$.
5. If $\underline{p} = \underline{i} + 8\underline{j} + 7\underline{k}$ and $\underline{q} = 3\underline{i} - 2\underline{j} + 5\underline{k}$, find $\underline{p} \cdot \underline{q}$.
6. Show that the vectors $\frac{1}{2}\underline{i} + \underline{j}$ and $2\underline{i} - \underline{j}$ are perpendicular.
7. The work done by a force $\underline{F}$ in moving a body through a displacement $\underline{r}$ is given by $\underline{F} \cdot \underline{r}$. Find the work done by the force $\underline{F} = 3\underline{i} + 7\underline{k}$ if it causes a body to move from the point with coordinates $(1, 1, 2)$ to the point $(7, 3, 5)$.
8. Find the angle between the vectors $\underline{i} - \underline{j} - \underline{k}$ and $2\underline{i} + \underline{j} + 2\underline{k}$.

Answers

1. -4.
2. $142.1°$,
3. This follows from the fact that $\cos\theta = 0$ since $\theta = 90°$.
4. $3a^2 - 10b^2$.
5. 22.
6. This follows from the scalar product being zero.
7. 39 units.
8. $101.1°$

5. Vectors and electrostatics

Electricity is important in several branches of engineering - not only in electrical or electronic engineering. For example the design of the electrostatic precipitator plates for cleaning the solid fuel power stations involves both mechanical engineering (structures and mechanical rapping systems for cleaning the plates) and electrostatics (to determine the electrical forces between solid particles and plates).

The following example and tasks relate to the electrostatic forces between particles. Electric charge is measured in coulombs (C). Charges can be either positive or negative.

The force between two charges

Let q_1 and q_2 be two charges in free space located at points P_1 and P_2. Then q_1 will experience a force due to the presence of q_2 and directed from P_2 towards P_1.
This force is of magnitude $K\,\dfrac{q_1q_2}{r^2}$ where r is the distance between P_1 and P_2 and K is a constant.
In vector notation this coulomb force (measured in newtons) can then be expressed as $\underline{F} = K\dfrac{q_1q_2}{r^2}\,\hat{\underline{r}}$ where $\hat{\underline{r}}$ is a unit vector directed from P_2 towards P_1.
The constant K is known to be $\dfrac{1}{4\pi\varepsilon_0}$ where $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1} (farads per metre).

The electric field

A unit charge located at a general point G will then experience a force $\dfrac{Kq_1}{r_1^2}\hat{\underline{r}}_1$ (where $\hat{\underline{r}}_1$ is the unit vector directed from P_1 towards G) due to a charge q_1 located at P_1. This is the electric field $\underline{E}$ newtons per coulomb (N C^{-1} or alternatively V m^{-1}) at G due to the presence of q_1.
For several point charges q_1 at P_1, q_2 at P_2 etc., the total electric field $\underline{E}$ at G is given by

$$\underline{E} = \frac{Kq_1}{r_1^2}\,\hat{\underline{r}}_1 + \frac{Kq_2}{r_2^2}\,\hat{\underline{r}}_2 + \dots$$

where $\hat{\underline{r}}_i$ is the unit vector directed from point P_i towards G.
From the definition of a unit vector we see that

$$\underline{E} = \frac{Kq_1}{r_1^2}\,\frac{\underline{r}_1}{|\underline{r}_1|} + \frac{Kq_2}{r_2^2}\,\frac{\underline{r}_2}{|\underline{r}_2|} + \dots = \frac{Kq_1}{|r_1|^3}\underline{r}_1 + \frac{Kq_2}{|\underline{r}_2|^3}\underline{r}_2 + \dots = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{|\underline{r}_1|^3}\underline{r}_1 + \frac{q_2}{|\underline{r}_2|^3}\underline{r}_2 + \dots\right]$$

where $\underline{r}_i$ is the vector directed from point P_i towards G, so that $\underline{r}_1 = \underline{OG} - \underline{OP}_1$ etc., where $\underline{OG}$ and $\underline{OP}_1$ are the position vectors of G and P_1 (see Figure 34).

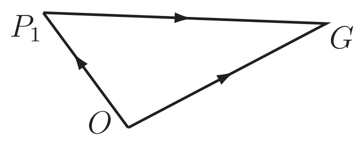

Figure 34

$$\underline{OP_1} + \underline{P_1G} = \underline{OG} \qquad\qquad \underline{P_1G} = \underline{OG} - \underline{OP_1}$$

The work done

The work done W (energy expended) in moving a charge q through a distance dS, in a direction given by the unit vector $\underline{S}/|\underline{S}|$, in an electric field $\underline{E}$ is (defined by)

$$W = -q\underline{E}.\text{d}\underline{S} \qquad (4)$$

where W is in joules.

Engineering Example 1

Field due to point charges

In free space, point charge $q_1 = 10\ n$C (1 nC $= 10^{-9}$C, i.e. a nanocoulomb) is at $P_1(0, -4, 0)$ and charge $q_2 = 20\ n$C is at $P_2 = (0, 0, 4)$.

[Note: Since the x-coordinate of both charges is zero, the problem is two-dimensional in the yz plane as shown in Figure 35.]

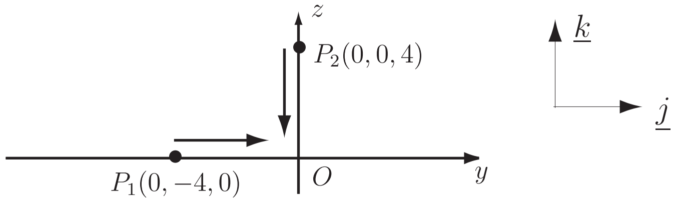

Figure 35

(a) Find the field at the origin $\underline{E}_{1,2}$ due to q_1 and q_2.

(b) Where should a third charge $q_3 = 30\ n$C be placed in the yz plane so that the total field due to q_1, q_2, q_3 is zero at the origin?

Solution

(a) Total field at the origin $\underline{E}_{1,2}$ = (field at origin due to charge at P_1) + (field at origin due to charge at P_2). Therefore

$$\underline{E}_{1,2} = \frac{10 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times 4^2}\underline{j} + \frac{20 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times 4^2}(-\underline{k}) = 5.617\underline{j} - 11.23\underline{k}$$

(The negative sign in front of the second term results from the fact that the direction from P_2 to O is in the $-z$ direction.)

(b) Suppose the third charge $q_3 = 30\ n$C is placed at $P_3(0, a, b)$. The field at the origin due to the third charge is

$$\underline{E}_3 = \frac{30 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (a^2 + b^2)} \times \frac{-(a\underline{j} + b\underline{k})}{(a^2 + b^2)^{1/2}},$$

where $\dfrac{a\underline{j} + b\underline{k}}{(a^2 + b^2)^{1/2}}$ is the unit vector in the direction from O to P_3

If the position of the third charge is such that the total field at the origin is zero, then $\underline{E}_3 = -\underline{E}_{1,2}$. There are two unknowns (a and b). We can write down two equations by considering the $\underline{j}$ and $\underline{k}$ directions.

Solution (contd.)

$$\underline{E}_3 = -269.6\left[\frac{a}{(a^2+b^2)^{3/2}}\underline{j} + \frac{b}{(a^2+b^2)^{3/2}}\underline{k}\right] \qquad \underline{E}_{1,2} = 5.617\underline{j} - 11.23\underline{k}$$

So

$$5.617 = 269.6 \times \frac{a}{(a^2+b^2)^{3/2}} \qquad (1)$$

$$-11.23 = 269.6 \times \frac{b}{(a^2+b^2)^{3/2}} \qquad (2)$$

So

$$\frac{a}{(a^2+b^2)^{3/2}} = 0.02083 \qquad (3)$$

$$\frac{b}{(a^2+b^2)^{3/2}} = -0.04165 \qquad (4)$$

Squaring and adding (3) and (4) gives $\dfrac{a^2+b^2}{(a^2+b^2)^3} = 0.002169$

So

$$(a^2+b^2) = 21.47 \qquad (5)$$

Substituting back from (5) into (1) and (2) gives $a = 2.07$ and $b = -4.14$, to 3 s.f.

Task

Eight point charges of 1 nC each are located at the corners of a cube in free space which is 1 m on each side (see Figure 36). Calculate $|\underline{E}|$ at

(a) the centre of the cube

(b) the centre of any face

(c) the centre of any edge.

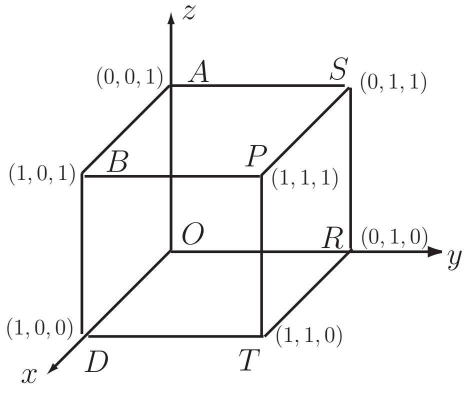

Figure 36

Your solution

Work the problem on a separate piece of paper but record here your main results and conclusions.

Answer

(a) The field at the centre of the cube is zero because of the symmetrical distribution of the charges.

(b) Because of the symmetrical nature of the problem it does not matter which face is chosen in order to find the magnitude of the field at the centre of a face. Suppose the chosen face has corners located at $P(1,1,1),\ T(1,1,0),\ R(0,1,0)$ and $S(0,1,1)$ then the centre (C) of this face can be seen from the diagram to be located at $C\left(\dfrac{1}{2},1,\dfrac{1}{2}\right)$.

The electric field at C due to the charges at the corners P,T,R and S will then be zero since the field vectors due to equal charges located at opposite corners of the square $PTRS$ cancel one another out. The field at C is then due to the equal charges located at the remaining four corners $(OABD)$ of the cube, and we note from the symmetry of the cube, that the distance of each of these corners from C will be the same. In particular the distance $OC = \sqrt{\left(\dfrac{1}{2}\right)^2 + 1^2 + \left(\dfrac{1}{2}\right)^2} = \sqrt{1.5}$ m. The electric field $\underline{E}$ at C due to the remaining charges can then be found using $\underline{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_1^4 \dfrac{q_i \cdot \underline{r}_i}{|\underline{r}_i|^3}$ where q_1 to q_4 are the equal charges (10^{-9} coulombs) and $\underline{r}_1$ to $\underline{r}_4$ are the vectors directed from the four corners, where the charges are located, towards C. In this case since $q_1 = 10^{-9}$ coulombs and $|\underline{r}_i| = \sqrt{1.5}$ for $i=1$ to $i=4$ we have

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{10^{-9}}{(1.5)^{3/2}}\left[\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4\right],$$

where $\underline{r}_1 = \underline{AC} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\underline{r}_2 = \underline{BC} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ etc.

Thus $\quad \underline{E} = \dfrac{1}{4\pi\varepsilon_0}\ \dfrac{10^{-9}}{(1.5)^{3/2}}\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

and $\quad |\underline{E}| = \dfrac{1}{\pi\varepsilon_0}\ \dfrac{10^{-9}}{(1.5)^{3/2}} = \dfrac{10^{-9}}{\pi \times 8.854 \times 10^{-12}(1.5)^{3/2}} = 19.57$ V m^{-1}

Answer

(c) Suppose the chosen edge to be used connects $A(0,0,1)$ to $B(1,0,1)$ then the centre point (G) will be located at $G\left(\frac{1}{2},0,1\right)$.

By symmetry the field at G due to the charges at A and B will be zero.

We note that the distances $DG,\ OG,\ PG$ and SG are all equal. In the case of OG we calculate by Pythagoras that this distance is $\sqrt{\left(\frac{1}{2}\right)^2+0^2+1^2}=\sqrt{1.25}$.

Similarly the distances TG and RG are equal to $\sqrt{2.25}$.

Using the result that $\underline{E}=\dfrac{1}{4\pi\varepsilon_0}\sum\dfrac{q_i\underline{r}_i}{|\underline{r}_i|^3}$ gives

$$\underline{E}=\frac{10^{-9}}{4\pi\varepsilon_0}\left[\frac{1}{(1.25)^{3/2}}\left\{\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\\0\\1\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}\\-1\\0\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\\-1\\0\end{bmatrix}\right\}\right.$$

$$\left.+\frac{1}{(2.25)^{3/2}}\left\{\begin{bmatrix}-\frac{1}{2}\\-1\\1\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\\-1\\1\end{bmatrix}\right\}\right]$$

$$=\frac{10^{-9}}{4\pi\varepsilon_0}\left[\frac{1}{(1.25)^{3/2}}\begin{bmatrix}0\\-2\\2\end{bmatrix}+\frac{1}{(2.25)^{3/2}}\begin{bmatrix}0\\-2\\2\end{bmatrix}\right]$$

$$=\frac{10^{-9}}{4\pi\varepsilon_0}\begin{bmatrix}0\\-2.02367\\2.02367\end{bmatrix}$$

Thus $|\underline{E}|=\dfrac{10^{-9}}{4\times\pi\times 8.854\times 10^{-12}}\sqrt{0^2+(-2.02367)^2+(2.02367)^2}$

$=25.72$ V m^{-1} (2 d.p.).

Task

If $\underline{E} = -50\underline{i} - 50\underline{j} + 30\underline{k}$ V m^{-1} where $\underline{i}$, $\underline{j}$ and $\underline{k}$ are unit vectors in the x, y and z directions respectively, find the differential amount of work done in moving a 2μC point charge a distance of 5 mm.

(a) From $P(1, 2, 3)$ towards $Q(2, 4, 1)$

(b) From $Q(2, 4, 1)$ towards $P(1, 2, 3)$

Your solution

Answer

(a) The work done in moving a 2μC charge through a distance of 5 mm towards Q is

$$\begin{aligned} W = -q\underline{E}.\underline{ds} &= -(2 \times 10^{-6})(5 \times 10^{-3})\underline{E}.\frac{PQ}{|\underline{PQ}|} \\ &= -10^{-8}(-50\underline{i} - 50\underline{j} + 30\underline{k}) \cdot \frac{(\underline{i} + 2\underline{j} - 2\underline{k})}{\sqrt{1^2 + 2^2 + (-2)^2}} \\ &= \frac{10^{-8}(50 + 100 + 60)}{3} = 7 \times 10^{-7} J \end{aligned}$$

(b) A similar calculation yields that the work done in moving the same charge through the same distance in the direction from Q to P is $W = -7 \times 10^{-7} J$