Lines and Planes

9.5

Introduction

Vectors are very convenient tools for analysing lines and planes in three dimensions. In this Section you will learn about direction ratios and direction cosines and then how to formulate the vector equation of a line and the vector equation of a plane. Direction ratios provide a convenient way of specifying the direction of a line in three dimensional space. Direction cosines are the cosines of the angles between a line and the coordinate axes. We begin this Section by showing how these quantities are calculated.

Prerequisites

Before starting this Section you should ...

- understand and be able to calculate the scalar product of two vectors
- understand and be able to calculate the vector product of two vectors

Learning Outcomes

On completion you should be able to ...

- obtain the vector equation of a line
- obtain the vector equation of a plane passing through a given point and which is perpendicular to a given vector
- obtain the vector equation of a plane which is a given distance from the origin and which is perpendicular to a given vector

1. The direction ratio and direction cosines

Consider the point $P(4, 5)$ and its position vector $4\underline{i} + 5\underline{j}$ shown in Figure 46.

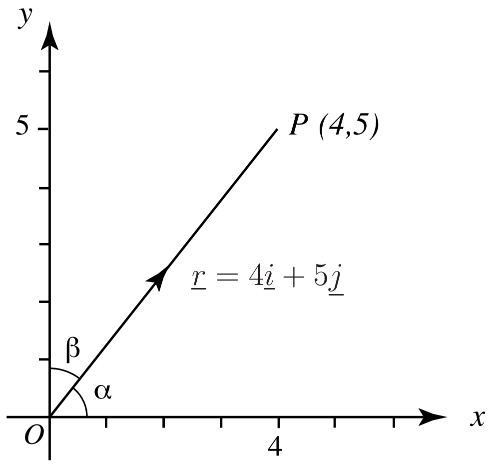

Figure 46

The **direction ratio** of the vector $\overrightarrow{OP}$ is defined to be 4:5. We can interpret this as stating that to move in the direction of the line OP we must move 4 units in the x direction for every 5 units in the y direction.

The **direction cosines** of the vector $\overrightarrow{OP}$ are the cosines of the angles between the vector and each of the axes. Specifically, referring to Figure 46 these are

$\cos\alpha$ and $\cos\beta$

Noting that the length of $\overrightarrow{OP}$ is $\sqrt{4^2 + 5^2} = \sqrt{41}$, we can write

$$\cos\alpha = \frac{4}{\sqrt{41}}, \qquad \cos\beta = \frac{5}{\sqrt{41}}.$$

It is conventional to label the direction cosines as ℓ and m so that

$$\ell = \frac{4}{\sqrt{41}}, \qquad m = \frac{5}{\sqrt{41}}.$$

More generally we have the following result:

Key Point 22

For **any** vector $\underline{r} = a\underline{i} + b\underline{j}$, its direction ratio is $a : b$.

Its direction cosines are

$$\ell = \frac{a}{\sqrt{a^2 + b^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2}}$$

Example 21

Point A has coordinates $(3,5)$, and point B has coordinates $(7,8)$.

(a) Write down the vector $\overrightarrow{AB}$.

(b) Find the direction ratio of the vector $\overrightarrow{AB}$.

(c) Find its direction cosines, ℓ and m.

(d) Show that $\ell^2 + m^2 = 1$.

Solution

(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = 4\underline{i} + 3\underline{j}$.

(b) The direction ratio of $\overrightarrow{AB}$ is therefore 4:3.

(c) The direction cosines are

$$\ell = \frac{4}{\sqrt{4^2+3^2}} = \frac{4}{5}, \qquad m = \frac{3}{\sqrt{4^2+3^2}} = \frac{3}{5}$$

(d)

$$\ell^2 + m^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

The final result in the previous Example is true in general:

Key Point 23

If ℓ and m are the direction cosines of a vector lying in the xy plane, then $\ell^2 + m^2 = 1$

Exercise

P and Q have coordinates $(-2,4)$ and $(7,8)$ respectively.

(a) Find the direction ratio of the vector $\overrightarrow{PQ}$ (b) Find the direction cosines of $\overrightarrow{PQ}$.

Answer

(a) $9:4$, (b) $\frac{9}{\sqrt{97}}, \frac{4}{\sqrt{97}}$.

2. Direction ratios and cosines in three dimensions

The concepts of direction ratio and direction cosines extend naturally to three dimensions. Consider Figure 47.

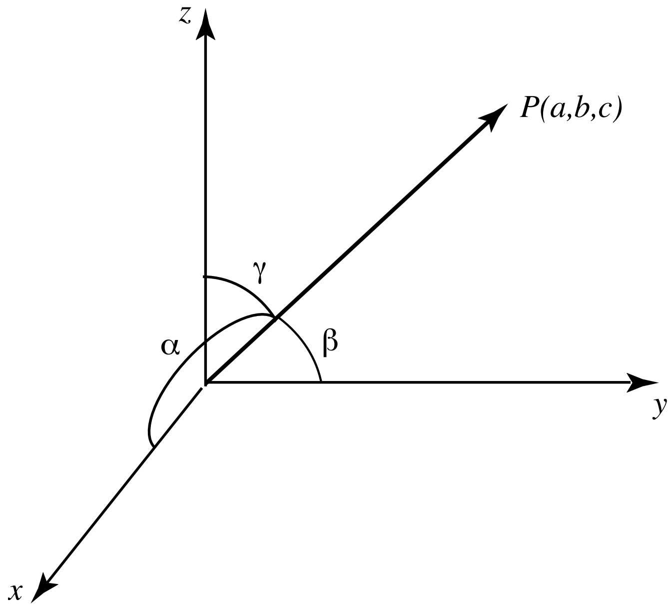

Figure 47

Given a vector $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ its direction ratios are $a : b : c$. This means that to move in the direction of the vector we must must move a units in the x direction and b units in the y direction for every c units in the z direction.

The direction cosines are the cosines of the angles between the vector and each of the axes. It is conventional to label direction cosines as ℓ, m and n and they are given by

$$\ell = \cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Wee have the following general result:

Key Point 24

For any vector $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ its direction ratios are $a : b : c$.

Its direction cosines are

$$\ell = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

where $\ell^2 + m^2 + n^2 = 1$

Exercises

1. Points A and B have position vectors $\underline{a} = -3\underline{i} + 2\underline{j} + 7\underline{k}$, and $\underline{b} = 3\underline{i} + 4\underline{j} - 5\underline{k}$ respectively. Find

   (a) $\overrightarrow{AB}$

   (b) $|\overrightarrow{AB}|$

   (c) The direction ratios of $\overrightarrow{AB}$

   (d) The direction cosines (ℓ, m, n) of $\overrightarrow{AB}$.

   (e) Show that $\ell^2 + m^2 + n^2 = 1$.

2. Find the direction ratios, the direction cosines and the angles that the vector $\overrightarrow{OP}$ makes with each of the axes when P is the point with coordinates $(2, 4, 3)$.

3. A line is inclined at $60°$ to the x axis and $45°$ to the y axis. Find its inclination to the z axis.

Answers

1. (a) $6\underline{i} + 2\underline{j} - 12\underline{k}$, (b) $\sqrt{184}$, (c) $6 : 2 : -12$, (d) $\dfrac{6}{\sqrt{184}}, \dfrac{2}{\sqrt{184}}, \dfrac{-12}{\sqrt{184}}$

2. 2:4:3; $\dfrac{2}{\sqrt{29}}, \dfrac{4}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}$; $68.2°$, $42.0°$, $56.1°$.

3. $60°$ or $120°$.

3. The vector equation of a line

Consider the straight line APB shown in Figure 48. This is a line in three-dimensional space.

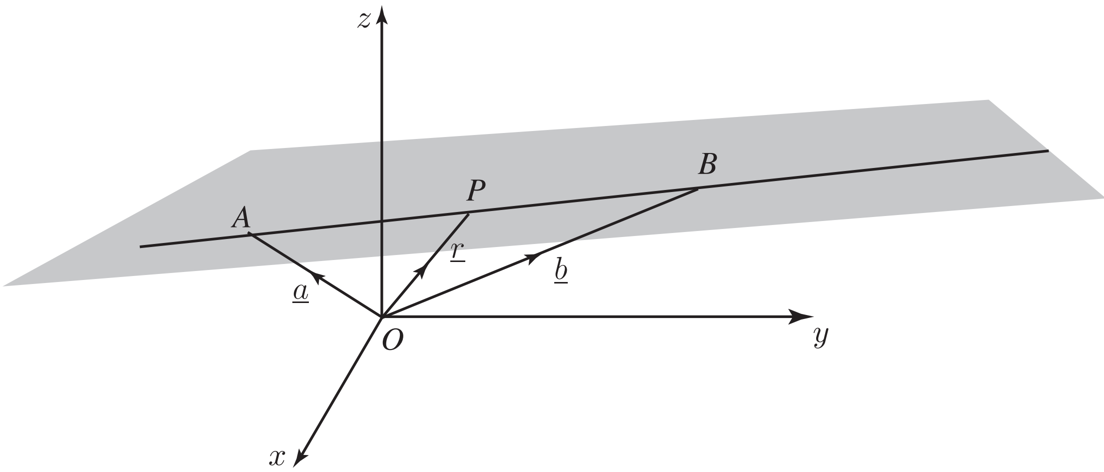

Figure 48

Points A and B are fixed and known points on the line, and have position vectors $\underline{a}$ and $\underline{b}$ respectively. Point P is any other arbitrary point on the line, and has position vector $\underline{r}$. Note that because $\overrightarrow{AB}$ and $\overrightarrow{AP}$ are parallel, $\overrightarrow{AP}$ is simply a scalar multiple of $\overrightarrow{AB}$, that is, $\overrightarrow{AP} = t\overrightarrow{AB}$ where t is a number.

Task

Referring to Figure 48, write down an expression for the vector $\overrightarrow{AB}$ in terms of $\underline{a}$ and $\underline{b}$.

Your solution

Answer

$\overrightarrow{AB} = \underline{b} - \underline{a}$

Task

Referring to Figure 48, use the triangle law for vector addition to find an expression for $\underline{r}$ in terms of $\underline{a}$, $\underline{b}$ and t, where $\overrightarrow{AP} = t\overrightarrow{AB}$.

Your solution

Answer

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

so that

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a}) \qquad \text{since } \overrightarrow{AP} = t\overrightarrow{AB}$$

The answer to the above Task, $\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$, is the **vector equation of the line** through A and B. It is a rule which gives the position vector $\underline{r}$ of a general point on the line in terms of the **given** vectors $\underline{a}, \underline{b}$. By varying the value of t we can move to any point on the line. For example, referring to Figure 48,

when $t = 0$, the equation gives $\underline{r} = \underline{a}$, which locates point A,

when $t = 1$, the equation gives $\underline{r} = \underline{b}$, which locates point B.

If $0 < t < 1$ the point P lies on the line between A and B. If $t > 1$ the point P lies on the line beyond B (to the right in the figure). If $t < 0$ the point P lies on the line beyond A (to the left in the figure).

Key Point 25

The **vector equation of the line** through points A and B with position vectors $\underline{a}$ and $\underline{b}$ is

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$$

Task

Write down the vector equation of the line which passes through the points with position vectors $\underline{a} = 3\underline{i} + 2\underline{j}$ and $\underline{b} = 7\underline{i} + 5\underline{j}$. Also express the equation in column vector form.

Your solution

Answer

$$\underline{b} - \underline{a} = (7\underline{i} + 5\underline{j}) - (3\underline{i} + 2\underline{j}) = 4\underline{i} + 3\underline{j}$$

The equation of the line is then

$$\begin{aligned} \underline{r} &= \underline{a} + t(\underline{b} - \underline{a}) \\ &= (3\underline{i} + 2\underline{j}) + t(4\underline{i} + 3\underline{j}) \end{aligned}$$

Using column vector notation we could write

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Task

Using column vector notation, write down the vector equation of the line which passes through the points with position vectors $\underline{a} = 5\underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} - 4\underline{k}$.

Your solution

Answer

Using column vector notation note that $\underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -7 \end{pmatrix}$

The equation of the line is then $\underline{r} = \underline{a} + t(\underline{b} - \underline{a}) = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 3 \\ -7 \end{pmatrix}$

Cartesian form

On occasions it is useful to convert the vector form of the equation of a straight line into Cartesian form. Suppose we write

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

then $\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$ implies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} = \begin{pmatrix} a_1 + t(b_1 - a_1) \\ a_2 + t(b_2 - a_2) \\ a_3 + t(b_3 - a_3) \end{pmatrix}$$

Equating the individual components we find

$$x = a_1 + t(b_1 - a_1), \qquad \text{or equivalently } t = \frac{x - a_1}{b_1 - a_1}$$

$$y = a_2 + t(b_2 - a_2), \qquad \text{or equivalently } t = \frac{y - a_2}{b_2 - a_2}$$

$$z = a_3 + t(b_3 - a_3), \qquad \text{or equivalently } t = \frac{z - a_3}{b_3 - a_3}$$

Each expression on the right is equal to t and so we can write

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$$

This gives the **Cartesian form** of the equations of the straight line which passes through the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3).

Key Point 26

The **Cartesian form** of the equation of the straight line which passes through the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) is

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$$

Example 22

(a) Write down the Cartesian form of the equation of the straight line which passes through the two points $(9, 3, -2)$ and $(4, 5, -1)$.

(b) State the equivalent vector equation.

Solution

(a)

$$\frac{x-9}{4-9} = \frac{y-3}{5-3} = \frac{z-(-2)}{-1-(-2)}$$

that is

$$\frac{x-9}{-5} = \frac{y-3}{2} = \frac{z+2}{1} \qquad \text{(Cartesian form)}$$

(b) The vector equation is

$$\begin{aligned} \underline{r} &= \underline{a} + t(\underline{b} - \underline{a}) \\ &= \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} + t\left(\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix}\right) \\ &= \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \end{aligned}$$

Exercises

1. (a) Write down the vector $\overrightarrow{AB}$ joining the points A and B with coordinates $(3, 2, 7)$ and $(-1, 2, 3)$ respectively.

   (b) Find the equation of the straight line through A and B.

2. Write down the vector equation of the line passing through the points with position vectors $\underline{p} = 3\underline{i} + 7\underline{j} - 2\underline{k}$ and $\underline{q} = -3\underline{i} + 2\underline{j} + 2\underline{k}$. Find also the Cartesian equation of this line.

3. Find the vector equation of the line passing through $(9, 1, 2)$ and which is parallel to the vector $(1, 1, 1)$.

Answers

1. (a) $-4\underline{i} - 4\underline{k}$. (b) $\underline{r} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ -4 \end{pmatrix}$.

2. $\underline{r} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix} + t\begin{pmatrix} -6 \\ -5 \\ 4 \end{pmatrix}$. Cartesian form $\dfrac{x-3}{-6} = \dfrac{y-7}{-5} = \dfrac{z+2}{4}$.

3. $\underline{r} = \begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4. The vector equation of a plane

Consider the plane shown in Figure 49.

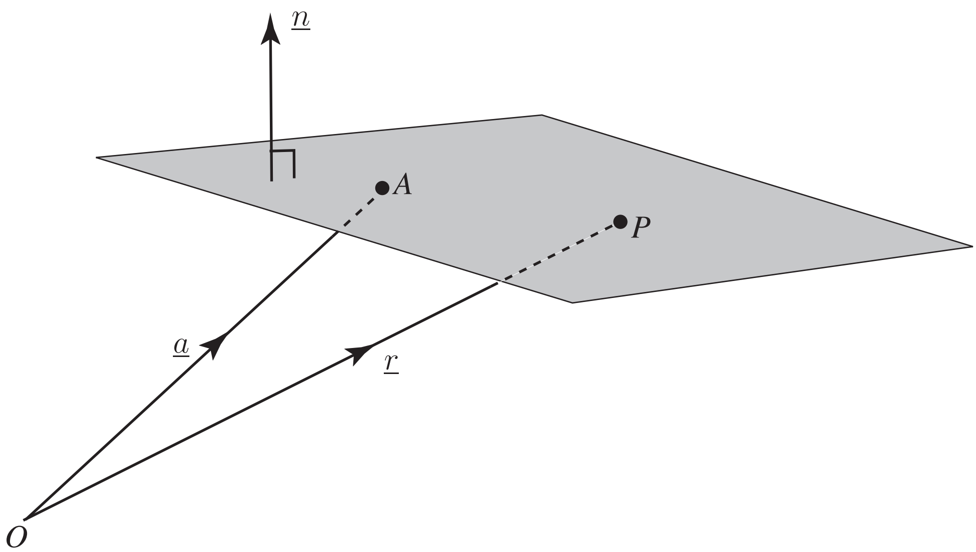

Figure 49

Suppose that A is a fixed point in the plane and has position vector $\underline{a}$. Suppose that P is any other arbitrary point in the plane with position vector $\underline{r}$. Clearly the vector $\overrightarrow{AP}$ lies in the plane.

Task Referring to Figure 49, find the vector $\overrightarrow{AP}$ in terms of $\underline{a}$ and $\underline{r}$.

Your solution

Answer

$\underline{r} - \underline{a}$

Also shown in Figure 49 is a vector which is perpendicular to the plane and denoted by $\underline{n}$.

Task What relationship exists between $\underline{n}$ and the vector $\overrightarrow{AP}$?

Hint: think about the scalar product:

Your solution

Answer

Because $\overrightarrow{AP}$ and $\underline{n}$ are perpendicular their scalar product must equal zero, that is

$$(\underline{r} - \underline{a}).\underline{n} = 0 \qquad \text{so that} \qquad \underline{r}.\underline{n} = \underline{a}.\underline{n}$$

The answer to the above Task, $\underline{r}.\underline{n} = \underline{a}.\underline{n}$, is the **equation of a plane**, written in vector form, passing through A and perpendicular to $\underline{n}$.

Key Point 27

A plane perpendicular to the vector $\underline{n}$ and passing through the point with position vector $\underline{a}$, has equation

$$\underline{r}.\underline{n} = \underline{a}.\underline{n}$$

In this formula it does not matter whether or not $\underline{n}$ is a unit vector.

If $\hat{\underline{n}}$ is a unit vector then $\underline{a}.\hat{\underline{n}}$ represents the perpendicular distance from the origin to the plane which we usually denote by d (for details of this see Section 9.3). Hence we can write

$$\underline{r}.\hat{\underline{n}} = d$$

This is the **equation of a plane**, written in vector form, with unit normal $\hat{\underline{n}}$ and which is a perpendicular distance d from O.

Key Point 28

A plane with unit normal $\hat{\underline{n}}$, which is a perpendicular distance d from O is given by

$$\underline{r}.\hat{\underline{n}} = d$$

Example 23

(a) Find the vector equation of the plane which passes through the point with position vector $3\underline{i} + 2\underline{j} + 5\underline{k}$ and which is perpendicular to $\underline{i} + \underline{k}$.

(b) Find the Cartesian equation of this plane.

Solution

(a) Using the previous results we can write down the equation

$$\underline{r}.(\underline{i} + \underline{k}) = (3\underline{i} + 2\underline{j} + 5\underline{k}).(\underline{i} + \underline{k}) = 3 + 5 = 8$$

(b) Writing $\underline{r}$ as $x\underline{i} + y\underline{j} + z\underline{k}$ we have the Cartesian form:

$$(x\underline{i} + y\underline{j} + z\underline{k}).(\underline{i} + \underline{k}) = 8$$

so that

$$x + z = 8$$

Task

(a) Find the vector equation of the plane through $(7, 3, -5)$ for which $\underline{n} = (1, 1, 1)$ is a vector normal to the plane.

(b) What is the distance of the plane from O?

Your solution

Answer

(a) Using the formula $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$ the equation of the plane is

$$\underline{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 7 \times 1 + 3 \times 1 - 5 \times 1 = 5$$

(b) The distance from the origin is $\underline{a}.\underline{\hat{n}} = \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \cdot \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \dfrac{5}{\sqrt{3}}$

Exercises

1. Find the equation of a plane which is normal to $8\underline{i} + 9\underline{j} + \underline{k}$ and which is a distance 1 from the origin. Give both vector and Cartesian forms.

2. Find the equation of a plane which passes through $(8, 1, 0)$ and which is normal to the vector $\underline{i} + 2\underline{j} - 3\underline{k}$.

3. What is the distance of the plane $\underline{r}. \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 5$ from the origin?

Answers

1. $\underline{r} \cdot \dfrac{1}{\sqrt{146}} \begin{pmatrix} 8 \\ 9 \\ 1 \end{pmatrix} = 1; \quad 8x + 9y + z = \sqrt{146}.$

2. $\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, that is $\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 10.$

3. $\dfrac{5}{\sqrt{14}}$