Integration by Substitution and using Partial Fractions

14.5

Introduction

The first technique described here involves making a substitution to simplify an integral. We let a new variable equal a complicated part of the function we are trying to integrate. Choosing the correct substitution often requires experience: don't worry if at first you cannot see an appropriate substitution. This skill develops with practice.
Often the technique of partial fractions can be used to write an algebraic fraction as the sum of simpler fractions. On occasions this means that we can then integrate a complicated algebraic fraction. We shall explore this approach in the second half of the Block.

Prerequisites

Before starting this Block you should ...

① be able to find a number of simple definite and indefinite integrals

② be able to use a table of integrals

③ be familiar with the technique of expressing an algebraic fraction as the sum of its partial fractions

Learning Outcomes

After completing this Block you should be able to ...

✓ make simple substitutions in order to find definite and indefinite integrals

✓ understand the technique used for evaluating integrals of the form $\int \frac{f'(x)}{f(x)}\mathrm{d}x$

✓ use partial fractions to express an algebraic fraction in a simpler form and hence integrate it

Learning Style

To achieve what is expected of you ...

☞ allocate sufficient study time

☞ briefly revise the prerequisite material

☞ attempt *every* guided exercise and most of the other exercises

1. Making a Substitution.

The technique described here involves making a substitution in order to simplify an integral. We let a new variable, u say, equal a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience: don't worry if at first you cannot see an appropriate substitution. This skill develops with practice. However, it is not simply a matter of changing the variable - care must be taken with the differential form $\mathrm{d}x$ as we shall see. The technique is illustrated in the following example.

Example Find the indefinite integral of $(3x+5)^6$; that is, find $\int (3x+5)^6 \mathrm{d}x$.

Solution

First look at the function we are trying to integrate: $(3x+5)^6$. It looks quite a complicated function to integrate. Suppose we introduce a new variable, u, such that $u = 3x+5$. Doing this means that the function we must integrate becomes u^6. Would you not agree that this looks a much simpler function to integrate than $(3x+5)^6$? There is a slight complication however. The new function of u must be integrated with respect to u and not with respect to x. This means that we must take care of the term $\mathrm{d}x$ correctly. From the substitution $u = 3x+5$ note, by differentiation, that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

It follows that we can write

$$\mathrm{d}x = \tfrac{1}{3}\mathrm{d}u$$

The required integral then becomes

$$\int (3x+5)^6 \mathrm{d}x = \int u^6 \,\frac{1}{3}\mathrm{d}u$$

The factor of $\frac{1}{3}$, being a constant, means that we can write

$$\begin{aligned}\int (3x+5)^6 \mathrm{d}x &= \frac{1}{3}\int u^6 \mathrm{d}u \\ &= \frac{1}{3}\left(\frac{u^7}{7}\right) + c \\ &= \frac{u^7}{21} + c\end{aligned}$$

To finish off we must rewrite this answer in terms of the original variable x and replace u by $3x+5$:

$$\int (3x+5)^6 \mathrm{d}x = \frac{(3x+5)^7}{21} + c$$

Now do this exercise

By making the substitution $u = \sin x$ find

$$\int \cos x \sin^2 x \,\mathrm{d}x$$

You are given the substitution $u = \sin x$. *Find* $\dfrac{\mathrm{d}u}{\mathrm{d}x}$:

Then make the substitution, simplify the result, and finally perform the integration. Answer

More exercises for you to try

Use a substitution to find

a) $\displaystyle\int (4x+1)^7 \mathrm{d}x$ b) $\displaystyle\int t^2 \sin(t^3+1)\mathrm{d}t$ (hint: let $u = t^3+1$)

Answer

2. Substitution and Definite Integrals

If you are dealing with definite integrals (ones with limits of integration) you must be particularly careful when you substitute. Consider the following example.

Example Find the definite integral $\int_2^3 t\sin(t^2)\mathrm{d}t$ by making the substitution $u = t^2$.

Solution

Note that if $u = t^2$ then $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ so that $\mathrm{d}t = \frac{\mathrm{d}u}{2t}$. We find

$$\begin{aligned}\int_{t=2}^{t=3} t\sin(t^2)\mathrm{d}t &= \int_{t=2}^{t=3} t\sin u \,\frac{\mathrm{d}u}{2t} \\ &= \frac{1}{2}\int_{t=2}^{t=3} \sin u\,\mathrm{d}u\end{aligned}$$

An important point to note is that the limits of integration are limits on the variable t, not u. To emphasise this they have been written explicitly as $t = 2$ and $t = 3$. When we integrate with respect to the variable u, the limits must be written in terms of u. From the substitution $u = t^2$, note that

$$\text{when } t = 2 \text{ then } u = 4 \quad \text{and when} \quad t = 3 \text{ then } u = 9$$

so the integral becomes

$$\begin{aligned}\frac{1}{2}\int_{u=4}^{u=9} \sin u\,\mathrm{d}u &= \frac{1}{2}[-\cos u]_4^9 \\ &= \frac{1}{2}(-\cos 9 + \cos 4) \\ &= 0.129\end{aligned}$$

More exercises for you to try

Use a substitution to find   a) $\displaystyle\int_1^2 (2x+3)^7\mathrm{d}x$,   b) $\displaystyle\int_0^1 3t^2 e^{t^3}\mathrm{d}t$.

Answer

3. Integrals Giving Rise to Logarithms

Example Find the indefinite integral of $3x^2 + 1$ divided by $x^3 + x + 2$; that is, find

$$\int \frac{3x^2+1}{x^3+x+2}\mathrm{d}x$$

Solution

Let us consider what happens when we make the substitution $z = x^3 + x + 2$. Note that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 + 1 \qquad \text{so that we can write} \qquad \mathrm{d}z = (3x^2+1)\mathrm{d}x$$

Then

$$\begin{aligned} \int \frac{3x^2+1}{x^3+x+2}\mathrm{d}x &= \int \frac{1}{z}\mathrm{d}z \\ &= \ln|z| + c \\ &= \ln|x^3+x+2| + c \end{aligned}$$

Note that in the last example, the numerator of the integrand is the derivative of the denominator. The result is the logarithm of the denominator. This is a special case of the following rule:

Key Point

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$$

Now do this exercise

Write down, purely by inspection, the following integrals:

a) $\displaystyle\int \frac{1}{x+1}\mathrm{d}x$, b) $\displaystyle\int \frac{2x}{x^2+8}\mathrm{d}x$, c) $\displaystyle\int \frac{1}{x-3}\mathrm{d}x$.

Hint: In each case the numerator of the integrand is the derivative of the denominator Answer

Now do this exercise

Evaluate the definite integral $\displaystyle\int_2^4 \frac{3t^2+2t}{t^3+t^2+1}\mathrm{d}t$.

Answer

Sometimes it is necessary to make slight adjustments to the integrand to obtain a form for which the previous rule is suitable. Consider the next example.

Example Find the indefinite integral $\displaystyle\int \frac{x^2}{x^3+1}\mathrm{d}x$.

Solution

In this example the derivative of the denominator is $3x^2$ whereas the numerator is just x^2. We adjust the numerator as follows:

$$\int \frac{x^2}{x^3+1}\mathrm{d}x = \frac{1}{3}\int \frac{3x^2}{x^3+1}\mathrm{d}x = \tfrac{1}{3}\ln|x^3+1| + c$$

Note that this sort of procedure is only possible because we can move constant factors through the integral sign. It would be wrong to try to move terms involving x in a similar way.

More exercises for you to try

1. Write down the result of finding the following integrals.

a) $\displaystyle\int \frac{1}{x}\mathrm{d}x$, b) $\displaystyle\int \frac{2t}{t^2+1}\mathrm{d}t$, c) $\displaystyle\int \frac{1}{2x+5}\mathrm{d}x$, d) $\displaystyle\int \frac{2}{3x-2}\mathrm{d}x$.

Answer

4. Integration using Partial Fractions

Sometimes expressions which at first sight look impossible to integrate using the techniques already met may in fact be integrated by first expressing them as simpler partial fractions, and then using the techniques described earlier in this Block. Consider the following Guided Exercise.

Try each part of this exercise

Part (a) Express $\displaystyle\frac{23-x}{(x-5)(x+4)}$ as the sum of its partial fractions. Hence find $\displaystyle\int \frac{23-x}{(x-5)(x+4)}\mathrm{d}x$

First produce the partial fractions.

(Hint: write the fraction in the form $\frac{A}{x-5}+\frac{B}{x+4}$ *then find* A, B*).*

Answer

Part (b) Now integrate each term separately.

$$\int \frac{23-x}{(x-5)(x+4)}\mathrm{d}x = \int \frac{A}{x-5}\mathrm{d}x + \int \frac{B}{x+4}\mathrm{d}x =$$

Answer

More exercises for you to try

By expressing the following in partial fractions evaluate the given integral:

1. $\displaystyle\int \frac{1}{x^3+x}dx$, 2. $\displaystyle\int \frac{13x-4}{6x^2-x-2}\mathrm{d}x$, 3. $\displaystyle\int \frac{1}{(x+1)(x-5)}\mathrm{d}x$

4. $\displaystyle\int \frac{2x}{(x-1)^2(x+1)}\mathrm{d}x$

Answer

End of Block 14.5

$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$ and $\int \cos x \sin^2 x \mathrm{d}x$ simplifies to $\int u^2 \mathrm{d}u$. The final answer is $\frac{1}{3}\sin^3 x + c$.

Back to the theory

a) $\dfrac{(4x+1)^8}{32}+c$ b) $-\dfrac{\cos(t^3+1)}{3}+c$

Back to the theory

a) 3.35886×10^5 b) 1.7183

Back to the theory

a) $\ln|x+1|+c$, b) $\ln|x^2+8|+c$, c) $\ln|x-3|+c$

Back to the theory

$[\ \ln|t^3+t^2+1|\ ]_2^4 = \ln 81 - \ln 13 = 1.83$

Back to the theory

1. a) $\ln|x| + c$, b) $\ln|t^2 + 1| + c$, c) $\frac{1}{2}\ln|2x + 5| + c$, d) $\frac{2}{3}\ln|3x - 2| + c$.

Back to the theory

$A = 2$, $B = -3$

Back to the theory

$2\ln|x-5| - 3\ln|x+4| + c$

Back to the theory

1. $\ln|x| - \frac{1}{2}\ln|x^2+1| + c$
2. $\frac{3}{2}\ln|2x+1| + \frac{2}{3}\ln|3x-2| + c$
3. $\frac{1}{6}\ln|x-5| - \frac{1}{6}\ln|x+1| + c.$
4. $-\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| - \frac{1}{x-1} + c.$

Back to the theory