Basics of z-Transform Theory

21.2

Introduction

In this Section, which is absolutely fundamental, we define what is meant by the z-transform of a sequence. We then obtain the z-transform of some important sequences and discuss useful properties of the transform.

Most of the results obtained are tabulated at the end of the Section.

The z-transform is the major mathematical tool for analysis in such areas as digital control and digital signal processing.

Prerequisites

Before starting this Section you should ...

- understand sigma (Σ) notation for summations
- be familiar with geometric series and the binomial theorem
- have studied basic complex number theory including complex exponentials

Learning Outcomes

On completion you should be able to ...

- define the z-transform of a sequence
- obtain the z-transform of simple sequences from the definition or from basic properties of the z-transform

1. The z-transform

If you have studied the Laplace transform either in a Mathematics course for Engineers and Scientists or have applied it in, for example, an analog control course you may recall that

1. the Laplace transform definition involves an **integral**
2. applying the Laplace transform to certain ordinary differential equations turns them into simpler (algebraic) equations
3. use of the Laplace transform gives rise to the basic concept of the **transfer function** of a continuous (or analog) system.

The z-transform plays a similar role for **discrete** systems, i.e. ones where **sequences** are involved, to that played by the Laplace transform for systems where the basic variable t is continuous. Specifically:

1. the z-transform definition involves a **summation**
2. the z-transform converts certain difference equations to algebraic equations
3. use of the z-transform gives rise to the concept of the transfer function of discrete (or digital) systems.

Key Point 1

Definition:

For a sequence $\{y_n\}$ the z-transform denoted by $Y(z)$ is given by the **infinite series**

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots = \sum_{n=0}^{\infty} y_n z^{-n} \qquad (1)$$

Notes:

1. The z-transform only involves the terms y_n, $n = 0, 1, 2, \ldots$ of the sequence. Terms $y_{-1}, y_{-2}, \ldots$ whether zero or non-zero, are not involved.
2. The infinite series in (1) **must converge** for $Y(z)$ to be defined as a precise function of z. We shall discuss this point further with specific examples shortly.
3. The precise significance of the quantity (strictly the 'variable') z need not concern us except to note that it is complex and, unlike n, is continuous.

Key Point 2

We use the notation $\mathbb{Z}\{y_n\} = Y(z)$ to mean that the z-transform of the sequence $\{y_n\}$ is $Y(z)$.

Less strictly one might write $\mathbb{Z}y_n = Y(z)$. Some texts use the notation $y_n \leftrightarrow Y(z)$ to denote that (the sequence) y_n and (the function) $Y(z)$ form a z-transform pair.

We shall also call $\{y_n\}$ the **inverse z-transform** of $Y(z)$ and write symbolically

$$\{y_n\} = \mathbb{Z}^{-1}Y(z).$$

2. Commonly used z-transforms

Unit impulse sequence (delta sequence)

This is a simple but important sequence denoted by δ_n and defined as

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, \ldots \end{cases}$$

The significance of the term 'unit impulse' is obvious from this definition.
By the definition (1) of the z-transform

$$\begin{aligned} \mathbb{Z}\{\delta_n\} &= 1 + 0z^{-1} + 0z^{-2} + \ldots \\ &= 1 \end{aligned}$$

If the single non-zero value is other than at $n = 0$ the calculation of the z-transform is equally simple. For example,

$$\delta_{n-3} = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

From (1) we obtain

$$\begin{aligned} \mathbb{Z}\{\delta_{n-3}\} &= 0 + 0z^{-1} + 0z^{-2} + z^{-3} + 0z^{-4} + \ldots \\ &= z^{-3} \end{aligned}$$

Task

Write down the definition of δ_{n-m} where m is any positive integer and obtain its z-transform.

Your solution

Answer

$$\delta_{n-m} = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{Z}\{\delta_{n-m}\} = z^{-m}$$

Key Point 3

$$\mathbb{Z}\{\delta_{n-m}\} = z^{-m} \qquad m = 0, 1, 2, \ldots$$

Unit step sequence

As we saw earlier in this Workbook the unit step sequence is

$$u_n = \begin{cases} 1 & n = 0, 1, 2, \ldots \\ 0 & n = -1, -2, -3, \ldots \end{cases}$$

Then, by the definition (1)

$$\mathbb{Z}\{u_n\} = 1 + 1z^{-1} + 1z^{-2} + \ldots$$

The infinite series here is a geometric series (with a constant ratio z^{-1} between successive terms). Hence the sum of the first N terms is

$$\begin{aligned} S_N &= 1 + z^{-1} + \ldots + z^{-(N-1)} \\ &= \frac{1 - z^{-N}}{1 - z^{-1}} \end{aligned}$$

As $N \to \infty \quad S_N \to \dfrac{1}{1 - z^{-1}}$ provided $|z^{-1}| < 1$

Hence, in what is called the **closed form** of this z-transform we have the result given in the following Key Point:

Key Point 4

$$\mathbb{Z}\{u_n\} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \equiv U(z) \text{ say}, \qquad |z^{-1}| < 1$$

The restriction that this result is only valid if $|z^{-1}| < 1$ or, equivalently $|z| > 1$ means that the position of the complex quantity z must lie outside the circle centre origin and of unit radius in an Argand diagram. This restriction is not too significant in elementary applications of the z-transform.

The geometric sequence $\{a^n\}$

Task

For any arbitrary constant a obtain the z-transform of the causal sequence

$$f_n = \begin{cases} 0 & n = -1, -2, -3, \ldots \\ a^n & n = 0, 1, 2, 3, \ldots \end{cases}$$

Your solution

Answer

We have, by the definition in Key Point 1,

$$F(z) = \mathbb{Z}\{f_n\} = 1 + az^{-1} + a^2z^{-2} + \ldots$$

which is a geometric series with common ratio az^{-1}. Hence, provided $|az^{-1}| < 1$, the closed form of the z-transform is

$$F(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}.$$

The z-transform of this sequence $\{a^n\}$, which is itself a geometric sequence is summarized in Key Point 5.

Key Point 5

$$\mathbb{Z}\{a^n\} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \qquad |z| > |a|.$$

Notice that if $a = 1$ we recover the result for the z-transform of the unit step sequence.

Task

Use Key Point 5 to write down the z-transform of the following causal sequences

(a) 2^n

(b) $(-1)^n$, the unit alternating sequence

(c) e^{-n}

(d) $e^{-\alpha n}$ where α is a constant.

Your solution

Answer

(a) Using $a = 2$ $\qquad \mathbb{Z}\{2^n\} = \dfrac{1}{1-2z^{-1}} = \dfrac{z}{z-2} \qquad |z| > 2$

(b) Using $a = -1$ $\qquad \mathbb{Z}\{(-1)^n\} = \dfrac{1}{1+z^{-1}} = \dfrac{z}{z+1} \qquad |z| > 1$

(c) Using $a = e^{-1}$ $\qquad \mathbb{Z}\{e^{-n}\} = \dfrac{z}{z-e^{-1}} \qquad |z| > e^{-1}$

(d) Using $a = e^{-\alpha}$ $\qquad \mathbb{Z}\{e^{-\alpha n}\} = \dfrac{z}{z-e^{-\alpha}} \qquad |z| > \mathrm{e}^{-\alpha}$

The basic z-transforms obtained have all been straightforwardly found from the definition in Key Point 1. To obtain further useful results we need a knowledge of some of the properties of z-transforms.

3. Linearity property and applications

Linearity property

This simple property states that if $\{v_n\}$ and $\{w_n\}$ have z-transforms $V(z)$ and $W(z)$ respectively then

$$\mathbb{Z}\{av_n + bw_n\} = aV(z) + bW(z)$$

for any constants a and b.
(In particular if $a = b = 1$ this property tells us that adding sequences corresponds to adding their z-transforms).

The proof of the linearity property is straightforward using obvious properties of the summation operation. By the z-transform definition:

$$\begin{aligned}
\mathbb{Z}\{av_n + bw_n\} &= \sum_{n=0}^{\infty}(av_n + bw_n)z^{-n} \\
&= \sum_{n=0}^{\infty}(av_n z^{-n} + bw_n z^{-n}) \\
&= a\sum_{n=0}^{\infty} v_n z^{-n} + b\sum_{n=0}^{\infty} w_n z^{-n} \\
&= aV(z) + bV(z)
\end{aligned}$$

We can now use the linearity property and the exponential sequence $\{e^{-\alpha n}\}$ to obtain the z-transforms of hyperbolic and of trigonometric sequences relatively easily. For example,

$$\sinh n = \frac{e^n - e^{-n}}{2}$$

Hence, by the linearity property,

$$\begin{aligned}
\mathbb{Z}\{\sinh n\} &= \frac{1}{2}\mathbb{Z}\{e^n\} - \frac{1}{2}\mathbb{Z}\{e^{-n}\} \\
&= \frac{1}{2}\left(\frac{z}{z-e} - \frac{z}{z-e^{-1}}\right) \\
&= \frac{z}{2}\left(\frac{z - e^{-1} - (z-e)}{z^2 - (e+e^{-1})z + 1}\right) \\
&= \frac{z}{2}\left(\frac{e - e^{-1}}{z^2 - (2\cosh 1)z + 1}\right) \\
&= \frac{z\sinh 1}{z^2 - 2z\cosh 1 + 1}
\end{aligned}$$

Using αn instead of n in this calculation, where α is a constant, we obtain

$$\mathbb{Z}\{\sinh \alpha n\} = \frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

Task

Using $\cosh \alpha n \equiv \dfrac{e^{\alpha n} + e^{-\alpha n}}{2}$ obtain the z-transform of the sequence $\{\cosh \alpha n\} = \{1,\ \cosh \alpha,\ \cosh 2\alpha, \ldots\}$

Your solution

Answer

We have, by linearity,

$$\begin{aligned}
\mathbb{Z}\{\cosh \alpha n\} &= \frac{1}{2}\mathbb{Z}\{e^{\alpha n}\} + \frac{1}{2}\mathbb{Z}\{e^{-\alpha n}\} \\
&= \frac{z}{2}\left(\frac{1}{z - e^{\alpha}} + \frac{1}{z - e^{-\alpha}}\right) \\
&= \frac{z}{2}\left(\frac{2z - (e^{\alpha} + e^{-\alpha})}{z^2 - 2z\cosh \alpha + 1}\right) \\
&= \frac{z^2 - z\cosh \alpha}{z^2 - 2z\cosh \alpha + 1}
\end{aligned}$$

Trigonometric sequences

If we use the result

$$\mathbb{Z}\{a^n\} = \frac{z}{z-a} \qquad |z| > |a|$$

with, respectively, $a = e^{\mathrm{i}\omega}$ and $a = e^{-\mathrm{i}\omega}$ where ω is a constant and i denotes $\sqrt{-1}$ we obtain

$$\mathbb{Z}\{e^{\mathrm{i}\omega n}\} = \frac{z}{z - e^{+\mathrm{i}\omega}} \qquad \mathbb{Z}\{e^{-\mathrm{i}\omega n}\} = \frac{z}{z - e^{-\mathrm{i}\omega}}$$

Hence, recalling from complex number theory that

$$\cos x = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}$$

we can state, using the linearity property, that

$$\begin{aligned}
\mathbb{Z}\{\cos\omega n\} &= \frac{1}{2}\mathbb{Z}\{e^{\mathrm{i}\omega n}\} + \frac{1}{2}\mathbb{Z}\{e^{-\mathrm{i}\omega n}\} \\
&= \frac{z}{2}\left(\frac{1}{z - e^{\mathrm{i}\omega}} + \frac{1}{z - e^{-\mathrm{i}\omega}}\right) \\
&= \frac{z}{2}\left(\frac{2z - (e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega})}{z^2 - (e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega})z + 1}\right) \\
&= \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}
\end{aligned}$$

(Note the similarity of the algebra here to that arising in the corresponding hyperbolic case. Note also the similarity of the results for $\mathbb{Z}\{\cosh\alpha n\}$ and $\mathbb{Z}\{\cos\omega n\}$.)

Task

By a similar procedure to that used above for $\mathbb{Z}\{\cos\omega n\}$ obtain $\mathbb{Z}\{\sin\omega n\}$.

Your solution

Answer
We have

$$\mathbb{Z}\{\sin \omega n\} = \frac{1}{2\mathrm{i}}\mathbb{Z}\{e^{\mathrm{i}\omega n}\} - \frac{1}{2\mathrm{i}}\mathbb{Z}\{e^{-\mathrm{i}\omega n}\} \qquad \text{(Don't miss the i factor here!)}$$

$$\begin{aligned}\therefore \qquad \mathbb{Z}\{\sin \omega n\} &= \frac{z}{2\mathrm{i}}\left(\frac{1}{z - e^{\mathrm{i}\omega}} - \frac{1}{z - e^{-\mathrm{i}\omega}}\right) \\ &= \frac{z}{2\mathrm{i}}\left(\frac{-e^{-\mathrm{i}\omega} + e^{\mathrm{i}\omega}}{z^2 - 2z\cos\omega + 1}\right) \\ &= \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}\end{aligned}$$

Key Point 6

$$\begin{aligned}\mathbb{Z}\{\cos \omega n\} &= \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1} \\ \mathbb{Z}\{\sin \omega n\} &= \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}\end{aligned}$$

Notice the same denominator in the two results in Key Point 6.

Key Point 7

$$\begin{aligned}\mathbb{Z}\{\cosh \alpha n\} &= \frac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1} \\ \mathbb{Z}\{\sinh \alpha n\} &= \frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}\end{aligned}$$

Again notice the denominators in Key Point 7. Compare these results with those for the two trigonometric sequences in Key Point 6.

Use Key Points 6 and 7 to write down the z-transforms of

(a) $\left\{\sin\dfrac{n}{2}\right\}$ (b) $\{\cos 3n\}$ (c) $\{\sinh 2n\}$ (d) $\{\cosh n\}$

Your solution

Answer

(a) $$\mathbb{Z}\left\{\sin\frac{n}{2}\right\} = \frac{z\sin\left(\frac{1}{2}\right)}{z^2-2z\cos\left(\frac{1}{2}\right)+1}$$

(b) $$\mathbb{Z}\{\cos 3n\} = \frac{z^2-z\cos 3}{z^2-2z\cos 3+1}$$

(c) $$\mathbb{Z}\{\sinh 2n\} = \frac{z\sinh 2}{z^2-2z\cosh 2+1}$$

(d) $$\mathbb{Z}\{\cosh n\} = \frac{z^2-z\cosh 1}{z^2-2z\cosh 1+1}$$

Task

Use the results for $\mathbb{Z}\{\cos\omega n\}$ and $\mathbb{Z}\{\sin\omega n\}$ in Key Point 6 to obtain the z-transforms of

(a) $\{\cos(n\pi)\}$ (b) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ (c) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}$

Write out the first few terms of each sequence.

Your solution

Answer

(a) With $\omega = \pi$

$$\mathbb{Z}\{\cos n\pi\} = \frac{z^2 - z\cos\pi}{z^2 - 2z\cos\pi + 1} = \frac{z^2 + z}{z^2 + 2z + 1} = \frac{z}{z+1}$$

$$\{\cos n\pi\} = \{1, -1, 1, -1, \ldots\} = \{(-1)^n\}$$

We have re-derived the z-transform of the unit alternating sequence. (See Task on page 17).

(b) With $\omega = \dfrac{\pi}{2}$

$$\mathbb{Z}\left\{\sin\frac{n\pi}{2}\right\} = \frac{z\sin\left(\frac{\pi}{2}\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1} = \frac{z}{z^2+1}$$

where $\left\{\sin\dfrac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, \ldots\}$

(c) With $\omega = \dfrac{\pi}{2}$ $\quad\mathbb{Z}\left\{\cos\dfrac{n\pi}{2}\right\} = \dfrac{z^2 - \cos\left(\frac{\pi}{2}\right)}{z^2+1} = \dfrac{z^2}{z^2+1}$

where $\left\{\cos\dfrac{n\pi}{2}\right\} = \{1, 0, -1, 0, 1, \ldots\}$

(These three results can also be readily obtained from the definition of the z-transform. Try!)

4. Further z-transform properties

We showed earlier that the results

$\mathbb{Z}\{v_n + w_n\} = V(z) + W(z)$ and similarly $\mathbb{Z}\{v_n - w_n\} = V(z) - W(z)$

follow from the linearity property.

You should be clear that there is **no** comparable result for the **product** of two sequences.

$\mathbb{Z}\{v_n w_n\}$ **is not equal to** $V(z)W(z)$

For two specific products of sequences however we can derive useful results.

Multiplication of a sequence by a^n

Suppose f_n is an arbitrary sequence with z-transform $F(z)$.
Consider the sequence $\{v_n\}$ where

$v_n = a^n f_n$ i.e. $\{v_0, v_1, v_2, \ldots\} = \{f_0, af_1, a^2 f_2, \ldots\}$

By the z-transform definition

$$\begin{aligned}
\mathbb{Z}\{v_n\} &= v_0 + v_1 z^{-1} + v_2 z^{-2} + \ldots \\
&= f_0 + a\ f_1 z^{-1} + a^2 f_2 z^{-2} + \ldots \\
&= \sum_{n=0}^{\infty} a^n f_n z^{-n} \\
&= \sum_{n=0}^{\infty} f_n \left(\frac{z}{a}\right)^{-n}
\end{aligned}$$

But $F(z) = \displaystyle\sum_{n=0}^{\infty} f_n z^{-n}$

Thus we have shown that $\mathbb{Z}\{a^n f_n\} = F\left(\dfrac{z}{a}\right)$

Key Point 8

$$\mathbb{Z}\{a^n f_n\} = F\left(\frac{z}{a}\right)$$

That is, multiplying a sequence $\{f_n\}$ by the sequence $\{a^n\}$ does not change the form of the z-transform $F(z)$. We merely replace z by $\dfrac{z}{a}$ in that transform.

For example, using Key Point 6 we have

$$\mathbb{Z}\{\cos n\} = \frac{z^2 - z\cos 1}{z^2 - 2z\cos 1 + 1}$$

So, replacing z by $\dfrac{z}{\left(\frac{1}{2}\right)} = 2z$,

$$\mathbb{Z}\left\{\left(\frac{1}{2}\right)^n \cos n\right\} = \frac{(2z)^2 - (2z)\cos 1}{(2z)^2 - 4z\cos 1 + 1}$$

Task

Using Key Point 8, write down the z-transform of the sequence $\{v_n\}$ where

$$v_n = e^{-2n}\sin 3n$$

Your solution

Answer

We have, $\quad \mathbb{Z}\{\sin 3n\} = \dfrac{z\sin 3}{z^2 - 2z\cos 3 + 1}$

so with $a = e^{-2}$ we replace z by $z\ e^{+2}$ to obtain

$$\begin{aligned}\mathbb{Z}\{v_n\} = \mathbb{Z}\{e^{-2n}\sin 3n\} &= \frac{ze^2\sin 3}{(ze^2)^2 - 2ze^2\cos 3 + 1} \\ &= \frac{ze^{-2}\sin 3}{z^2 - 2ze^{-2}\cos 3 + e^{-4}}\end{aligned}$$

Task

Using the property just discussed write down the z-transform of the sequence $\{w_n\}$ where

$$w_n = e^{-\alpha n}\cos\omega n$$

Your solution

Answer

We have, $\quad \mathbb{Z}\{\cos\omega n\} = \dfrac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$

So replacing z by ze^{α} we obtain

$$\begin{aligned}\mathbb{Z}\{w_n\} = \mathbb{Z}\{e^{-\alpha n}\cos\omega n\} &= \frac{(ze^{\alpha})^2 - ze^{\alpha}\cos\omega}{(ze^{\alpha})^2 - 2ze^{\alpha}\cos\omega + 1}\\ &= \frac{z^2 - ze^{-\alpha}\cos\omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}\end{aligned}$$

Key Point 9

$$\begin{aligned}\mathbb{Z}\{e^{-\alpha n}\cos\omega n\} &= \frac{z^2 - ze^{-\alpha}\cos\omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}\\ \mathbb{Z}\{e^{-\alpha n}\sin\omega n\} &= \frac{ze^{-\alpha}\sin\omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}\end{aligned}$$

Note the same denominator in each case.

Multiplication of a sequence by n

An important sequence whose z-transform we have not yet obtained is the **unit ramp** sequence $\{r_n\}$:

$$r_n = \begin{cases} 0 & n = -1, -2, -3, \ldots \\ n & n = 0, 1, 2, \ldots \end{cases}$$

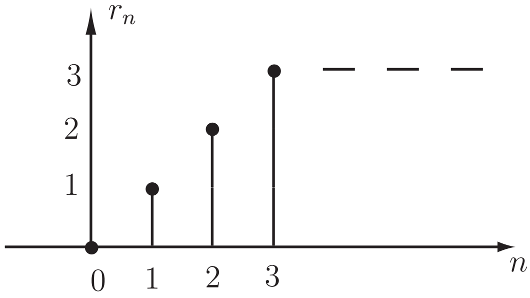


Figure 5

Figure 5 clearly suggests the nomenclature 'ramp'.

We shall attempt to use the z-transform of $\{r_n\}$ from the definition:

$$\mathbb{Z}\{r_n\} = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + \ldots$$

This is not a geometric series but we can write

$$\begin{aligned} z^{-1} + 2z^{-2} + 3z^{-3} &= z^{-1}(1 + 2z^{-1} + 3z^{-2} + \ldots) \\ &= z^{-1}(1 - z^{-1})^{-2} \qquad |z^{-1}| < 1 \end{aligned}$$

where we have used the binomial theorem (HELM 16.3) .
Hence

$$\begin{aligned} \mathbb{Z}\{r_n\} = \mathbb{Z}\{n\} &= \frac{1}{z\left(1 - \frac{1}{z}\right)^2} \\ &= \frac{z}{(z-1)^2} \qquad |z| > 1 \end{aligned}$$

Key Point 10

The z-transform of the unit ramp sequence is

$$\mathbb{Z}\{r_n\} = \frac{z}{(z-1)^2} = R(z) \quad \text{(say)}$$

Recall now that the unit step sequence has z-transform $\mathbb{Z}\{u_n\} = \dfrac{z}{(z-1)} = U(z)$ (say) which is the subject of the next Task.

Task

Obtain the derivative of $U(z)=\dfrac{z}{(z-1)}$ with respect to z.

Your solution

Answer

We have, using the quotient rule of differentiation:

$$\frac{dU}{dz}=\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{(z-1)1-(z)(1)}{(z-1)^2}$$

$$= \frac{-1}{(z-1)^2}$$

We also know that

$$R(z)=\frac{z}{(z-1)^2}=(-z)\left(-\frac{1}{(z-1)^2}\right)=-z\frac{dU}{dz} \qquad (3)$$

Also, if we compare the sequences

$$u_n=\{0,0,1,1,1,1,\ldots\}$$
$$\uparrow$$
$$r_n=\{0,0,0,1,2,3,\ldots\}$$
$$\uparrow$$

we see that $r_n = n\ u_n$, (4)

so from (3) and (4) we conclude that $\mathbb{Z}\{n\ u_n\} = -z\dfrac{dU}{dz}$

Now let us consider the problem more generally.

Let f_n be an arbitrary sequence with z-transform $F(z)$:

$$F(z)=f_0+f_1z^{-1}+f_2z^{-2}+f_3z^{-3}+\ldots=\sum_{n=0}^{\infty}f_nz^{-n}$$

We differentiate both sides with respect to the variable z, doing this term-by-term on the right-hand side. Thus

$$\begin{aligned}\frac{dF}{dz} &= -f_1z^{-2} - 2f_2z^{-3} - 3f_3z^{-4} - \ldots = \sum_{n=1}^{\infty}(-n)f_nz^{-n-1} \\ &= -z^{-1}(f_1z^{-1} + 2f_2z^{-2} + 3f_3z^{-3} + \ldots) = -z^{-1}\sum_{n=1}^{\infty} n\ f_nz^{-n}\end{aligned}$$

But the bracketed term is the z-transform of the sequence

$$\{n\ f_n\} = \{0, f_1, 2f_2, 3f_3, \ldots\}$$

Thus if $F(z) = \mathbb{Z}\{f_n\}$ we have shown that

$$\frac{dF}{dz} = -z^{-1}\mathbb{Z}\{n\ f_n\} \quad \text{or} \quad \mathbb{Z}\{n\ f_n\} = -z\frac{dF}{dz}$$

We have already (equations (3) and (4) above) demonstrated this result for the case $f_n = u_n$.

Key Point 11

If $\mathbb{Z}\{f_n\} = F(z)$ then $\mathbb{Z}\{n\ f_n\} = -z\dfrac{dF}{dz}$

Task

By differentiating the z-transform $R(z)$ of the unit ramp sequence obtain the z-transform of the causal sequence $\{n^2\}$.

Your solution

Answer
We have

$$\mathbb{Z}\{n\} = \frac{z}{(z-1)^2}$$

so

$$\mathbb{Z}\{n^2\} = \mathbb{Z}\{n.n\} = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

By the quotient rule

$$\begin{aligned}\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) &= \frac{(z-1)^2-(z)(2)(z-1)}{(z-1)^4}\\ &= \frac{z-1-2z}{(z-1)^3} = \frac{-1-z}{(z-1)^3}\end{aligned}$$

Multiplying by $-z$ we obtain

$$\mathbb{Z}\{n^2\} = \frac{z+z^2}{(z-1)^3} = \frac{z(1+z)}{(z-1)^3}$$

Clearly this process can be continued to obtain the transforms of $\{n^3\},\ \{n^4\},\ldots$ etc.

5. Shifting properties of the z-transform

In this subsection we consider perhaps the most important properties of the z-transform. These properties relate the z-transform $Y(z)$ of a sequence $\{y_n\}$ to the z-transforms of

(i) **right shifted** or **delayed sequences** $\{y_{n-1}\}\{y_{n-2}\}$ etc.
(ii) **left shifted** or **advanced sequences** $\{y_{n+1}\},\{y_{n+2}\}$ etc.

The results obtained, formally called shift theorems, are vital in enabling us to solve certain types of difference equation and are also invaluable in the analysis of digital systems of various types.

Right shift theorems

Let $\{v_n\} = \{y_{n-1}\}$ i.e. the terms of the sequence $\{v_n\}$ are the same as those of $\{y_n\}$ but shifted one place to the right. The z-transforms are, by definition,

$$\begin{aligned}Y(z) &= y_0 + y_1z^{-1} + y_2z^{-2} + y_jz^{-3} + \ldots\\ V(z) &= v_0 + v_1z^{-1} + v_2z^{-2} + v_3z^{-3} + \ldots\\ &= y_{-1} + y_0z^{-1} + y_1z^{-2} + y_2z^{-3} + \ldots\\ &= y_{-1} + z^{-1}(y_0 + y_1z^{-1} + y_2z^{-2} + \ldots)\end{aligned}$$

i.e.

$$V(z) = \mathbb{Z}\{y_{n-1}\} = y_{-1} + z^{-1}Y(z)$$

Task

Obtain the z-transform of the sequence $\{w_n\} = \{y_{n-2}\}$ using the method illustrated above.

Your solution

Answer

The z-transform of $\{w_n\}$ is $\quad W(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3} + \ldots \quad$ or, since $w_n = y_{n-2}$,

$$\begin{aligned} W(z) &= y_{-2} + y_{-1}z^{-1} + y_0 z^{-2} + y_1 z^{-3} + \ldots \\ &= y_{-2} + y_{-1}z^{-1} + z^{-2}(y_0 + y_1 z^{-1} + \ldots) \end{aligned}$$

i.e. $W(z) = \mathbb{Z}\{y_{n-2}\} = y_{-2} + y_{-1}z^{-1} + z^{-2}Y(z)$

Clearly, we could proceed in a similar way to obtains a general result for $\mathbb{Z}\{y_{n-m}\}$ where m is any positive integer. The result is

$$\mathbb{Z}\{y_{n-m}\} = y_{-m} + y_{-m+1}z^{-1} + \ldots + y_{-1}z^{-m+1} + z^{-m}Y(z)$$

For the particular case of causal sequences (where $y_{-1} = y_{-2} = \ldots = 0$) these results are particularly simple:

$$\left.\begin{aligned} \mathbb{Z}\{y_{n-1}\} &= z^{-1}Y(z) \\ \mathbb{Z}\{y_{n-2}\} &= z^{-2}Y(z) \\ \\ \mathbb{Z}\{y_{n-m}\} &= z^{-m}Y(z) \end{aligned}\right\} \text{(causal systems only)}$$

You may recall from earlier in this Workbook that in a digital system we represented the right shift operation symbolically in the following way:

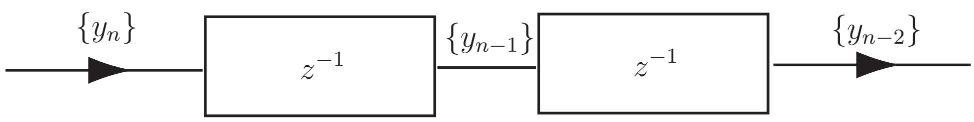


Figure 6

The significance of the z^{-1} factor inside the rectangles should now be clearer. If we replace the 'input' and 'output' sequences by their z-transforms:

$$\mathbb{Z}\{y_n\} = Y(z) \qquad \mathbb{Z}\{y_{n-1}\} = z^{-1}Y(z)$$

it is evident that in the z-transform 'domain' the shift becomes a multiplication by the factor z^{-1}.
N.B. This discussion applies strictly only to causal sequences.

Notational point:

A causal sequence is sometimes written as $y_n u_n$ where u_n is the unit step sequence

$$u_n = \begin{cases} 0 & n = -1, -2, \ldots \\ 1 & n = 0, 1, 2, \ldots \end{cases}$$

The right shift theorem is then written, for a causal sequence,

$$\mathbb{Z}\{y_{n-m}u_{n-m}\} = z^{-m}Y(z)$$

Examples

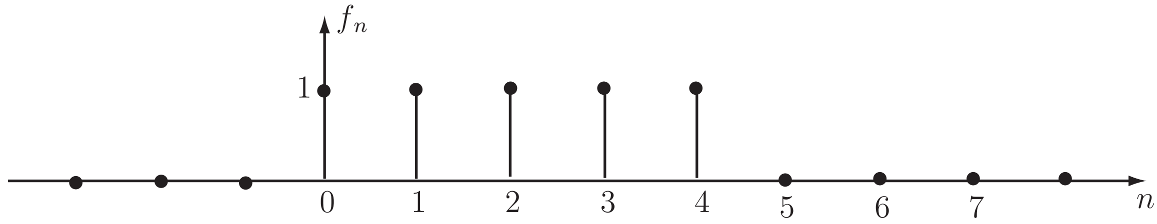
Recall that the z-transform of the causal sequence $\{a^n\}$ is $\dfrac{z}{z-a}$. It follows, from the right shift theorems that

(i) $\mathbb{Z}\{a^{n-1}\} = \mathbb{Z}\{\underset{\uparrow}{0}, 1, a, a^2, \ldots\} = \dfrac{zz^{-1}}{z-a} = \dfrac{1}{z-a}$

(ii) $\mathbb{Z}\{a^{n-2}\} = \mathbb{Z}\{\underset{\uparrow}{0}, 0, 1, a, a^2, \ldots\} = \dfrac{z^{-1}}{z-a} = \dfrac{1}{z(z-a)}$

Task

Write the following sequence f_n as a difference of two unit step sequences. Hence obtain its z-transform.



Your solution

Answer

Since $\{u_n\} = \begin{cases} 1 & n = 0, 1, 2, \ldots \\ 0 & n = -1, -2, \ldots \end{cases}$

and $\{u_{n-5}\} = \begin{cases} 1 & n = 5, 6, 7, \ldots \\ 0 & \text{otherwise} \end{cases}$

it follows that

$$f_n = u_n - u_{n-5}$$

Hence $\qquad F(z) = \dfrac{z}{z-1} - \dfrac{z^{-5}z}{z-1} = \dfrac{z - z^{-4}}{z-1}$

Left shift theorems

Recall that the sequences $\{y_{n+1}\},\ \{y_{n+2}\} \ldots$ denote the sequences obtained by shifting the sequence $\{y_n\}$ by $1, 2, \ldots$ units to the left respectively. Thus, since $Y(z) = \mathbb{Z}\{y_n\} = y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots$ then

$$\begin{aligned} \mathbb{Z}\{y_{n+1}\} &= y_1 + y_2 z^{-1} + y_3 z^{-2} + \ldots \\ &= y_1 + z(y_2 z^{-2} + y_3 z^{-3} + \ldots) \end{aligned}$$

The term in brackets is the z-transform of the unshifted sequence $\{y_n\}$ apart from its first two terms: thus

$$\mathbb{Z}\{y_{n+1}\} = y_1 + z(Y(z) - y_0 - y_1 z^{-1})$$

$$\therefore \qquad Z\{y_{n+1}\} = zY(z) - zy_0$$

Task

Obtain the z-transform of the sequence $\{y_{n+2}\}$ using the method illustrated above.

Your solution

Answer

$$\begin{aligned}
\mathbb{Z}\{y_{n+2}\} &= y_2 + y_3 z^{-1} + y_4 z^{-2} + \dots \\
&= y_2 + z^2(y_3 z^{-3} + y_4 z^{-4} + \dots) \\
&= y_2 + z^2(Y(z) - y_0 - y_1 z^{-1} - y_2 z^{-2})
\end{aligned}$$

$$\therefore \qquad \mathbb{Z}\{y_{n+2}\} = z^2 Y(z) - z^2 y_0 - z y_1$$

These left shift theorems have simple forms in special cases:

if $y_0 = 0 \qquad \mathbb{Z}\{y_{n+1}\} = z\ Y(z)$

if $y_0 = y_1 = 0 \qquad \mathbb{Z}\{y_{n+2}\} = z^2 Y(z)$

if $y_0 = y_1 = \dots y_{m-1} = 0 \qquad \mathbb{Z}\{y_{n+m}\} = z^m Y(z)$

Key Point 12

The **right shift theorems** or delay theorems are:

$$\begin{aligned}
\mathbb{Z}\{y_{n-1}\} &= y_{-1} + z^{-1} Y(z) \\
\mathbb{Z}\{y_{n-2}\} &= y_{-2} + y_{-1} z^{-1} + z^{-2} Y(z) \\
\vdots \qquad & \vdots \qquad \vdots \qquad \vdots \\
\mathbb{Z}\{y_{n-m}\} &= y_{-m} + y_{-m+1} z^{-1} + \dots + y_{-1} z^{-m+1} + z^{-m} Y(z)
\end{aligned}$$

The **left shift theorems** or advance theorems are:

$$\begin{aligned}
\mathbb{Z}\{y_{n+1}\} &= zY(z) - z y_0 \\
\mathbb{Z}\{y_{n+2}\} &= z^2 Y(z) - z^2 y_0 - z y_1 \\
\vdots \qquad & \vdots \\
\mathbb{Z}\{y_{n-m}\} &= z^m Y(z) - z^m y_0 - z^{m-1} y_1 - \dots - z y_{m-1}
\end{aligned}$$

Note carefully the occurrence of **positive** powers of z in the **left** shift theorems and of **negative** powers of z in the **right** shift theorems.

Table 1: z-transforms

f_n	$F(z)$	Name
δ_n	1	unit impulse
δ_{n-m}	z^{-m}	
u_n	$\dfrac{z}{z-1}$	unit step sequence
a^n	$\dfrac{z}{z-a}$	geometric sequence
$e^{\alpha n}$	$\dfrac{z}{z-e^{\alpha}}$	
$\sinh \alpha n$	$\dfrac{z \sinh \alpha}{z^2 - 2z\cosh\alpha + 1}$	
$\cosh \alpha n$	$\dfrac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}$	
$\sin \omega n$	$\dfrac{z \sin \omega}{z^2 - 2z\cos\omega + 1}$	
$\cos \omega n$	$\dfrac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$	
$e^{-\alpha n} \sin \omega n$	$\dfrac{ze^{-\alpha} \sin \omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}$	
$e^{-\alpha n} \cos \omega n$	$\dfrac{z^2 - ze^{-\alpha} \cos \omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}$	
n	$\dfrac{z}{(z-1)^2}$	ramp sequence
n^2	$\dfrac{z(z+1)}{(z-1)^3}$	
n^3	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	
$a^n f_n$	$F\left(\dfrac{z}{a}\right)$	
$n\ f_n$	$-z\dfrac{dF}{dz}$	

This table has been copied to the back of this Workbook (page 96) for convenience.