Standard Complex Functions

26.3

Introduction

In this Section we examine some of the standard functions of the calculus applied to functions of a complex variable. Note the similarities to and differences from their equivalents in real variable calculus.

Prerequisites

Before starting this Section you should ...

- understand the concept of a function of a complex variable and its derivative
- be familiar with the Cauchy-Riemann equations

Learning Outcomes

On completion you should be able to ...

- apply the standard functions of a complex variable discussed in this Section

1. Standard functions of a complex variable

The functions which we have considered so far have mostly been built from powers of z. We consider other functions here.

The exponential function

Using Euler's relation we are led to define

$$e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y).$$

From this definition we can show readily that when $y = 0$ then e^z reduces to e^x, as it should.
If, as usual, we express w in real and imaginary parts then: $w = e^z = u + iv$ so that
$u = e^x \cos y,\ v = e^x \sin y$. Then

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Thus by the Cauchy-Riemann equations, **e^z is analytic everywhere**. It can be shown from the definition that if $f(z) = e^z$ then $f'(z) = e^z$, as expected.

Task

By calculating $|e^z|^2$ show that $|e^z| = e^x$.

Your solution

Answer

$|e^z|^2 = |e^x \cos y + ie^x \sin y|^2 = (e^x \cos y)^2 + (e^x \sin y)^2 = (e^x)^2(\cos^2 y + \sin^2 y) = (e^x)^2.$

Therefore $|e^z| = e^x$.

Example 7

Find $\arg(e^z)$.

Solution

If $\theta = \arg(e^z) = \arg(e^x(\cos y + i\sin y))$ then $\tan\theta = \dfrac{e^x \sin y}{e^x \cos y} = \tan y$. Hence $\arg(e^z) = y$.

Example 8

Find the solutions (for z) of the equation $\mathrm{e}^z = \mathrm{i}$

Solution

To find the solutions of the equation $\mathrm{e}^z = \mathrm{i}$ first write i as $0+1\mathrm{i}$ so that, equating real and imaginary parts of $\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = 0 + 1\mathrm{i}$ gives , $\mathrm{e}^x \cos y = 0$ and $\mathrm{e}^x \sin y = 1$.

Therefore $\cos y = 0$, which implies $y = \dfrac{\pi}{2} + k\pi$, where k is an integer. Then, using this we see that $\sin y = \pm 1$. But e^x must be positive, so that $\sin y = +1$ and $\mathrm{e}^x = 1$. This last equation has just one solution, $x = 0$. In order that $\sin y = 1$ we deduce that k must be even. Finally we have the complete solution to $\mathrm{e}^z = \mathrm{i}$, namely:

$$z = \left(\frac{\pi}{2} + k\pi\right)\mathrm{i},\ k \text{ an even integer.}$$

Task

Obtain all the solutions to $\mathrm{e}^z = -1$.

First find equations involving $\mathrm{e}^x \cos y$ and $\mathrm{e}^x \sin y$:

Your solution

Answer

As a first step to solving the equation $\mathrm{e}^z = -1$ obtain expressions for $\mathrm{e}^x \cos y$ and $\mathrm{e}^x \sin y$ from $\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = -1 + 0\mathrm{i}$. Hence $\mathrm{e}^x \cos y = -1$, $\mathrm{e}^x \sin y = 0$.

Now using the expression for $\sin y$ deduce possible values for y and hence from the first equation in $\cos y$ select the values of y satisfying both equations and deduce the form of the solutions for z:

Your solution

Answer

The two equations we have to solve are: $\mathrm{e}^x \cos y = -1$, $\mathrm{e}^x \sin y = 0$. Since $\mathrm{e}^x \neq 0$ we deduce $\sin y = 0$ so that $y = k\pi$, where k is an integer. Then $\cos y = \pm 1$ (depending as k is even or odd). But $\mathrm{e}^x \neq -1$ so $\mathrm{e}^x = 1$ leading to the only possible solution for x: $x = 0$. Then, from the second relation: $\cos y = -1$ so k must be an odd integer. Finally, $z = k\pi\mathrm{i}$ where k is an odd integer. Note the interesting result that if $z = 0 + \pi\mathrm{i}$ then $x = 0$, $y = \pi$ and $\mathrm{e}^z = 1(\cos\pi + \mathrm{i}\sin\pi) = -1$. Hence $\mathrm{e}^{\mathrm{i}\pi} = -1$, a remarkable equation relating fundamental numbers of mathematics in one relation.

Trigonometric functions

We denote the complex counterparts of the real trigonometric functions $\cos x$ and $\sin x$ by $\cos z$ and $\sin z$ and we define these functions by the relations:

$$\cos z \equiv \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}), \quad \sin z \equiv \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}).$$

These definitions are consistent with the definitions (Euler's relations) used for $\cos x$ and $\sin x$.

Other trigonometric functions can be defined in a way which parallels real variable functions. For example,

$$\tan z \equiv \frac{\sin z}{\cos z}.$$

Note that

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left\{\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})\right\} = \frac{1}{2\mathrm{i}}(\mathrm{i}\mathrm{e}^{\mathrm{i}z} + \mathrm{i}\mathrm{e}^{-\mathrm{i}z}) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) = \cos z.$$

Task

Show that $\dfrac{d}{dz}(\cos z) = -\sin z$.

Your solution

Answer

$$\begin{aligned}\frac{d}{dz}(\cos z) &= \frac{d}{dz}\left\{\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})\right\} \\ &= \frac{\mathrm{i}}{2}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) = -\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) = -\sin z.\end{aligned}$$

Among other useful relationships are

$$\begin{aligned}\sin^2 z + \cos^2 z &= -\frac{1}{4}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})^2 + \frac{1}{4}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})^2 \\ &= \frac{1}{4}(-\mathrm{e}^{2\mathrm{i}z} + 2 - \mathrm{e}^{-2\mathrm{i}z} + \mathrm{e}^{2\mathrm{i}z} + 2 + \mathrm{e}^{-2\mathrm{i}z}) = \frac{1}{4}\cdot 4 = 1.\end{aligned}$$

Also, using standard trigonometric expansions:

$$\begin{aligned}\sin z = \sin(x + \mathsf{i}y) = \sin x \cos \mathsf{i}y + \cos x \sin \mathsf{i}y &= \sin x \left(\frac{\mathsf{e}^{-y} + \mathsf{e}^{y}}{2}\right) + \cos x \left(\frac{\mathsf{e}^{-y} - \mathsf{e}^{y}}{2\mathsf{i}}\right) \\ &= \sin x \cosh y - \frac{1}{\mathsf{i}} \cos x \sinh y \\ &= \sin x \cosh y + \mathsf{i} \cos x \sinh y.\end{aligned}$$

Task

Show that $\cos z = \cos x \cosh y - \mathsf{i} \sin x \sinh y$.

Your solution

Answer

$$\begin{aligned}\cos z = \cos(x + \mathsf{i}y) = \cos x \cos \mathsf{i}y - \sin x \sin \mathsf{i}y &= \cos x \left(\frac{\mathsf{e}^{-y} + \mathsf{e}^{y}}{2}\right) - \sin x \left(\frac{\mathsf{e}^{-y} - \mathsf{e}^{y}}{2\mathsf{i}}\right) \\ &= \cos x \cosh y + \frac{1}{\mathsf{i}} \sin x \sinh y \\ &= \cos x \cosh y - \mathsf{i} \sin x \sinh y\end{aligned}$$

Hyperbolic functions

In an obvious extension from their real variable counterparts we define functions $\cosh z$ and $\sinh z$ by the relations:

$$\cosh z = \frac{1}{2}(\mathsf{e}^{z} + \mathsf{e}^{-z}), \qquad \sinh z = \frac{1}{2}(\mathsf{e}^{z} - \mathsf{e}^{-z}).$$

Note that $\dfrac{d}{dz}(\sinh z) = \dfrac{1}{2}\dfrac{d}{dz}(\mathsf{e}^{z} - \mathsf{e}^{-z}) = \dfrac{1}{2}(\mathsf{e}^{z} + \mathsf{e}^{-z}) = \cosh z$.

Task

Determine $\dfrac{d}{dz}(\cosh z)$.

Your solution

Answer

$\dfrac{d}{dz}(\cosh z) = \dfrac{1}{2}\dfrac{d}{dz}(\mathsf{e}^{z} + \mathsf{e}^{-z}) = \dfrac{1}{2}(\mathsf{e}^{z} - \mathsf{e}^{-z}) = \sinh z.$

Other relationships parallel those for trigonometric functions. For example it can be shown that

$$\cosh z = \cosh x \cos y + \mathsf{i} \sinh x \sin y \qquad \text{and} \qquad \sinh z = \sinh x \cos y + \mathsf{i} \cosh x \sin y$$

These relationships can be deduced from the general relations between trigonometric and hyperbolic functions (can you prove these?):

$$\cosh \mathsf{i}z = \cos z \qquad \text{and} \qquad \sinh \mathsf{i}z = \mathsf{i} \sin z$$

Example 9

Show that $\cosh^2 z - \sinh^2 z = 1$.

Solution

$$\begin{aligned} \cosh^2 z &= \frac{1}{4}(\mathsf{e}^{z} + \mathsf{e}^{-z})^2 = \frac{1}{4}(\mathsf{e}^{2z} + 2 + \mathsf{e}^{-2z}) \\ \sinh^2 z &= \frac{1}{4}(\mathsf{e}^{z} - \mathsf{e}^{-z})^2 = \frac{1}{4}(\mathsf{e}^{2z} - 2 + \mathsf{e}^{-2z}) \\ \therefore \quad \cosh^2 z &- \sinh^2 z = \frac{1}{4}(2+2) = 1. \end{aligned}$$

Alternatively since $\cosh \mathsf{i}z = \cos z$ then $\cosh z = \cos \mathsf{i}z$ and since $\sinh \mathsf{i}z = \mathsf{i} \sin z$ it follows that $\sinh z = -\mathsf{i} \sin \mathsf{i}z$ so that

$$\cosh^2 z - \sinh^2 z = \cos^2 \mathsf{i}z + \sin^2 \mathsf{i}z = 1$$

Logarithmic function

Since the exponential function is one-to-one it possesses an inverse function, which we call $\ln z$. If $w = u + \mathrm{i}v$ is a complex number such that $\mathrm{e}^w = z$ then the logarithm function is defined through the statement: $w = \ln z$. To see what this means it will be convenient to express the complex number z in exponential form as discussed in HELM 10.3: $z = r\mathrm{e}^{\mathrm{i}\theta}$ and so

$$w = u + \mathrm{i}v = \ln(r\mathrm{e}^{\mathrm{i}\theta}) = \ln r + \mathrm{i}\theta.$$

Therefore $u = \ln r = \ln|z|$ and $v = \theta$. However $\mathrm{e}^{\mathrm{i}(\theta+2k\pi)} = \mathrm{e}^{\mathrm{i}\theta}.\mathrm{e}^{2k\pi\mathrm{i}} = \mathrm{e}^{\mathrm{i}\theta}.1 = \mathrm{e}^{\mathrm{i}\theta}$ for integer k. This means that we must be more general and say that $v = \theta + 2k\pi$, k integer. If we take $k = 0$ and confine v to the interval $-\pi < v \leq \pi$, the corresponding value of w is called the **principal value** of $\ln z$ and is written $\mathrm{Ln}(z)$.

In general, to each value of $z \neq 0$ there are an infinite number of values of $\ln z$, each with the same real part. These values are partitioned into **branches** of range 2π by considering in turn $k = 0$, $k = \pm 1$, $k = \pm 2$ etc. Each branch is defined on the whole $z-$plane with the exception of the point $z = 0$. On each branch the function $\ln z$ is analytic with derivative $\dfrac{1}{z}$ **except** along the negative real axis (and at the origin). Figure 6 represents the situation schematically.

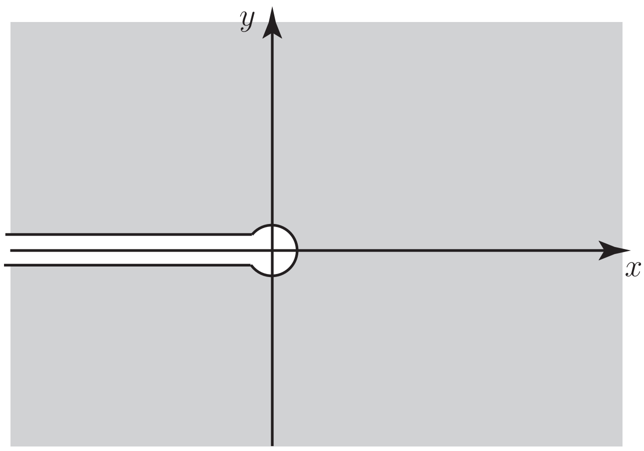

Figure 6

The familiar properties of a logarithm apply to $\ln z$, **except** that in the case of $\mathrm{Ln}(z)$ we have to adjust the argument by a multiple of 2π to comply with $-\pi < \arg(\mathrm{Ln}(z)) \leq \pi$
For example

(a) $\ln(1+\mathrm{i}) = \ln\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right) = \ln\sqrt{2} + \mathrm{i}\left(\frac{\pi}{4} + 2k\pi\right)$
$$= \frac{1}{2}\ln 2 + \mathrm{i}\left(\frac{\pi}{4} + 2k\pi\right).$$

(b) $\mathrm{Ln}(1+\mathrm{i}) = \dfrac{1}{2}\ln 2 + \mathrm{i}\dfrac{\pi}{4}.$

(c) If $\ln z = 1 - \mathrm{i}\pi$ then $z = \mathrm{e}^{1-\mathrm{i}\pi} = \mathrm{e}^1.\mathrm{e}^{-\mathrm{i}\pi} = -e.$

Task Find (a) $\ln(1-\mathsf{i})$ (b) $\mathsf{Ln}(1-\mathsf{i})$ (c) z when $\ln z = 1 + \mathsf{i}\pi$

Your solution

Answer

(a) $\ln(1-\mathsf{i}) = \ln\left(\sqrt{2}\mathsf{e}^{-\mathsf{i}\frac{\pi}{4}}\right) = \ln\sqrt{2} + \mathsf{i}\left(-\dfrac{\pi}{4} + 2k\pi\right) = \dfrac{1}{2}\ln 2 + \left(-\dfrac{\pi}{4} + 2k\pi\right).$

(b) $\mathsf{Ln}(1-\mathsf{i}) = \dfrac{1}{2}\ln 2 - \mathsf{i}\dfrac{\pi}{4}.$

(c) $z = \mathsf{e}^{1+\mathsf{i}\pi} = \mathsf{e}^1.\mathsf{e}^{\mathsf{i}\pi} = -e.$

Exercises

1. Obtain all the solutions to $\mathsf{e}^z = 1$.
2. Show that $1 + \tan^2 z \equiv \sec^2 z$
3. Show that $\cosh^2 z + \sinh^2 z \equiv \cosh 2z$
4. Find $\ln(\sqrt{3}+\mathsf{i})$, $\mathsf{Ln}(\sqrt{3}+\mathsf{i})$.
5. Find z when $\ln z = 2 + \pi\mathsf{i}$

Answers

1. $\mathsf{e}^x \cos y = 1$ and $\mathsf{e}^x \sin y = 0$ $\quad\therefore\quad \sin y = 0$ and $y = k\pi$ where k is an integer.

   Then $\cos y = \pm 1$ and since $\mathsf{e}^x > 0$ we take $\cos y = 1$ and $\mathsf{e}^x = 1$ so that $x = 0$. Then $\cos y = 1$ and k is an even integer. $\quad\therefore\quad z = 2k\pi\mathsf{i}$ for k integer.

2. $\tan z = \dfrac{1}{\mathsf{i}}\left(\dfrac{\mathsf{e}^{\mathsf{i}z} - \mathsf{e}^{-\mathsf{i}z}}{\mathsf{e}^{\mathsf{i}z} + \mathsf{e}^{-\mathsf{i}z}}\right)$

   $$1 + \tan^2 z = 1 - \frac{\mathsf{e}^{2\mathsf{i}z} + \mathsf{e}^{-2\mathsf{i}z} - 2}{\mathsf{e}^{2\mathsf{i}z} + \mathsf{e}^{-2\mathsf{i}z} + 2} = \frac{4}{\mathsf{e}^{2\mathsf{i}z} + \mathsf{e}^{-2\mathsf{i}z} + 2} = \frac{2^2}{(\mathsf{e}^{\mathsf{i}z} + \mathsf{e}^{-\mathsf{i}z})^2} = \frac{1}{\cos^2 z} = \sec^2 z.$$

3. $\cosh^2 z + \sinh^2 z = \dfrac{1}{4}(\mathsf{e}^{2z} + 2 + \mathsf{e}^{-2z}) + \dfrac{1}{4}(\mathsf{e}^{2z} - 2 + \mathsf{e}^{-2z}) = \dfrac{1}{2}(\mathsf{e}^{2z} + \mathsf{e}^{-2z}) = \cosh 2z.$

4. $\ln(\sqrt{3}+1) = \ln\sqrt{5} + \mathsf{i}(\frac{\pi}{6} + 2k\pi) = \frac{1}{2}\ln 5 + \mathsf{i}(\frac{\pi}{6} + 2k\pi).$ $\qquad \mathsf{Ln}(\sqrt{3}+\mathsf{i}) = \frac{1}{2}\ln 5 + \mathsf{i}\frac{\pi}{6}.$

5. If $\ln z = 2 + \pi\mathsf{i}$ then $z = \mathsf{e}^{2+\pi i} = \mathsf{e}^2\mathsf{e}^{\mathsf{i}\pi} = -\mathsf{e}^2.$