Elliptic PDEs

33.2

Introduction

In HELM 32.4 and 32.5, we saw methods of obtaining numerical solutions to Parabolic and Hyperbolic partial differential equations (PDEs). Another class of PDEs are the Elliptic type, and these usually model time-independent situations. In this Section we will concentrate on two particularly important Elliptic type PDEs: Laplace's equation and Poisson's equation.

Prerequisites

Before starting this Section you should ...

- familiarise yourself with difference methods for approximating second derivatives (HELM 31.3)
- revise the Jacobi and Gauss-Seidel methods from (HELM 30.5)

Learning Outcomes

On completion you should be able to ...

- obtain simple approximate solutions of certain elliptic equations

1. Elliptic equations

Consider a region R (for example, a rectangle) in the xy-plane. We might pose the following boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= f(x,y) && \text{a given function, in } R \\ u &= g && \text{a given function, on the boundary of } R \end{aligned}$$

- if $f = 0$ everywhere, then the PDE is called **Laplace's equation**
- if f is non-zero somewhere in R then the PDE is called **Poisson's equation**

Laplace's equation models a huge range of physical situations. It is used by coastal engineers to approximate the motion of the sea; it is used to model electric potential; it can give an approximation to heat distribution in certain steady state problems. The list goes on and on. The generalisation to Poisson's equation opens up further application areas, but for our purposes in this Section we will concentrate on how to solve the equation, rather than on how it is applied.

2. A five point stencil

The approach we shall use is to approximate the two second derivatives using central differences. First we need some notation for our numerical solution, and we shall re-use some of the ideas seen in HELM 32.4 and HELM 32.5. We divide the x-axis up into subintervals of width δx and the y-axis into subintervals of width δy.

There is a simplification available to us now that was not possible in HELM 32. Here, the two independent variables (x and y) both measure distance (in HELM 32 we had x measuring distance and t measuring time) and there is no reason to suppose that one direction is more important than another, so we may choose the subintervals δx and δy to be equal.

Key Point 4

In deriving numerical solutions to elliptic PDEs we use equal steps in the x and y directions. That is, we take

$$\delta x = \delta y = h \quad \text{(say)}$$

So the idea is to approximate the second derivatives in the familiar way:

$$u_{xx} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}, \qquad u_{yy} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

We will write our numerical approximation as

$$\underset{\substack{\uparrow \\ \text{numerical} \\ \text{approximation}}}{u_{i,j}} \approx \underset{\substack{\uparrow \\ \text{exact (i.e. unknown) solution} \\ \text{evaluated at } x = i \times h,\ y = j \times h}}{\underbrace{u(i\,h\ ,\ j\,h)}}$$

Key Point 5

We use **subscripts** on u to relate to space variables. For Elliptic PDEs both of the independent variables measure distance and so we have *two* subscripts.

Key Point 6

If there is no danger of ambiguity we may omit the comma from the subscript. That is,

$$u_{i,j} \quad \text{may be written} \quad u_{ij} \quad \text{and} \quad f_{i,j} \quad \text{may be written} \quad f_{ij}$$

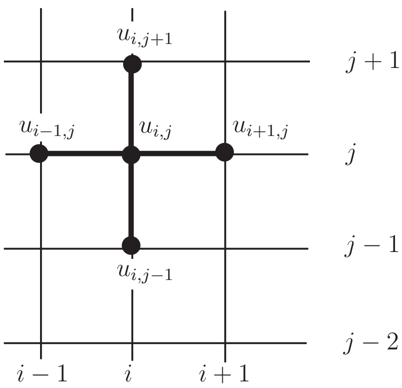
Given all of this preamble we can now write down a difference equation which approximates the partial differential equation:

$$\underbrace{\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}}_{\approx\, u_{xx}} + \underbrace{\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}}_{\approx\, u_{yy}} = \underset{\substack{\uparrow \\ \text{notation for } f(ih,\,jh)}}{f_{i,j}}$$

Rearranging this gives

$$-4u_{i,j}+u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}=h^2 f_{i,j}$$

This equation defines a **five-point stencil** approximating the PDE. The following diagram shows the stencil.

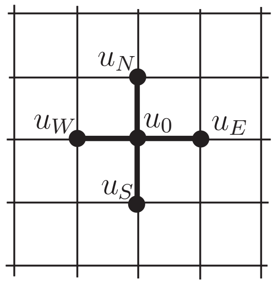


The idea in an implementation of this stencil is to centre the cross-shape on each i, j node where we want to find u. This guarantees that we will end up with the same number of equations as unknowns. An example of this approach will follow shortly, but first we note other ways of writing down the five-point stencil.

As the diagram above shows, the stencil involves a centre point and four additional points each corresponding to one of the points of the compass. It is this observation which has led to a simplified version of the mathematical expression and the diagram. The symbolic stencil can be written

$$-4u_0 + u_E + u_W + u_N + u_S = h^2 f_0,$$

where a subscript 0 corresponds to the centre of the stencil and other subscripts correspond to compass points (North, South, East, West) in the obvious way. The diagram becomes



and we reinterpret the local "$0, N, S, E, W$" positions each time we move the stencil on the global grid.

Another way of writing the stencil is as follows:

	1	
1	-4	1
	1	

This latest version has the advantage of showing the values of the coefficients used in approximating $u_{xx} + u_{yy}$.

We summarise in Key Point 7 the main idea using the notation established above.

Key Point 7

The **five-point stencil** used to approximate the partial differential equation

$$u_{xx} + u_{yy} = f(x, y)$$

gives rise to the difference equation

$$-4u_0 + u_E + u_W + u_N + u_S = h^2 f_0$$

Example 6

Consider the boundary value problem

$$u_{xx} + u_{yy} = 0 \qquad \text{in the square } 0 < x < 1, \;\; 0 < y < 1$$
$$u = x^2 y \qquad \text{on the boundary.}$$

Use $h = \frac{1}{3}$ and formulate a system of simultaneous equations for the 4 unknowns.

Solution

In the diagram on the right we see a schematic of the square in the xy plane. The numbers correspond to boundary data where the numerical grid intersects that boundary. The (as yet unknown) numerical approximations are shown in the positions where they approximate $u(x, y)$.

$$\begin{array}{ccccccc} y & \uparrow & & & & & \\ & 0 & \frac{1}{9} & \frac{4}{9} & 1 & & \\ & 0 & u_{12} & u_{22} & \frac{2}{3} & & \\ & 0 & u_{11} & u_{21} & \frac{1}{3} & & \\ & 0 & 0 & 0 & 0 & \rightarrow & x \end{array}$$

The numerical stencil in this case is $-4u_0 + u_E + u_W + u_N + u_S = 0$ and we centre this at each of the places where u is sought. There are four such places in this example:

$$\begin{array}{rccccccccccc}
\text{bottom left:} & -4u_{11} & + & u_{21} & + & 0 & + & u_{12} & + & 0 & = & 0 \\
\text{bottom right:} & -4u_{21} & + & \frac{1}{3} & + & u_{11} & + & u_{22} & + & 0 & = & 0 \\
\text{top left:} & -4u_{12} & + & u_{22} & + & 0 & + & \frac{1}{9} & + & u_{11} & = & 0 \\
\text{top right:} & -4u_{22} & + & \frac{2}{3} & + & u_{12} & + & \frac{4}{9} & + & u_{21} & = & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & \text{Centre} & & \text{East} & & \text{West} & & \text{North} & & \text{South} & &
\end{array}$$

This is a system of equations in the four unknowns which may be written

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = - \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{9} \\ \frac{10}{9} \end{pmatrix}$$

It is now a (simple, in theory) matter of solving the system to obtain the numerical approximation to u.

It turns out that the solution to the system of equations is $u_{11} = \frac{1}{12}$, $u_{21} = \frac{7}{36}$, $u_{12} = \frac{5}{36}$ and $u_{22} = \frac{13}{36}$. These values are, to four decimal places, 0.0833, 0.1944, 0.1389 and 0.3611, respectively. We will say more later about how to solve the system of equations, but first there is a Task to help consolidate what we have covered so far.

Task

Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= -2 \qquad \text{in the square } 0 < x < 1, \quad 0 < y < 1 \\ u &= xy \qquad \text{on the boundary.} \end{aligned}$$

Use $h = \frac{1}{3}$ and hence formulate a system of simultaneous equations for the four unknowns.

Your solution

Answer

In the diagram on the right we see a schematic of the square in the xy plane. The numbers correspond to boundary data where the numerical grid intersects that boundary. The (as yet unknown) numerical approximations are shown in the positions where they approximate $u(x,y)$.

$$\begin{array}{lllll} y\uparrow & & & & \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 & \\ 0 & u_{12} & u_{22} & \frac{2}{3} & \\ 0 & u_{11} & u_{21} & \frac{1}{3} & \\ 0 & 0 & 0 & 0 & \rightarrow x \end{array}$$

The numerical stencil in this case is

$$-4u_0 + u_E + u_W + u_N + u_S = h^2 f_0 = (\frac{1}{3})^2 \times (-2) = -\frac{2}{9}$$

and we centre this at each of the places where u is sought. In this Example there are four such places:

$$\begin{array}{rccccccccccc} \text{bottom left:} & -4u_{11} & + & u_{21} & + & 0 & + & u_{12} & + & 0 & = & -\frac{2}{9} \\ \text{bottom right:} & -4u_{21} & + & \frac{1}{3} & + & u_{11} & + & u_{22} & + & 0 & = & -\frac{2}{9} \\ \text{top left:} & -4u_{12} & + & u_{22} & + & 0 & + & \frac{1}{3} & + & u_{11} & = & -\frac{2}{9} \\ \text{top right:} & -4u_{22} & + & \frac{2}{3} & + & u_{12} & + & \frac{2}{3} & + & u_{21} & = & -\frac{2}{9} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ & \text{Centre} & & \text{East} & & \text{West} & & \text{North} & & \text{South} & & \end{array}$$

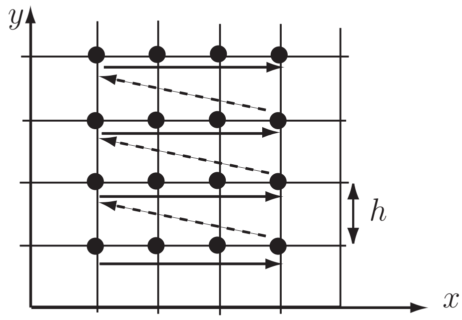
This is a system of equations in the four unknowns and it may be written

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = - \begin{pmatrix} \frac{2}{9} \\ \frac{5}{9} \\ \frac{5}{9} \\ \frac{14}{9} \end{pmatrix}$$

3. Systems of equations

In order to obtain accurate results over a large number of interior points, we need to decrease h compared to the values used in the Examples above.

The diagram below shows a case where 5 steps are used in each direction on a square domain. It follows that there will be $4 \times 4 = 16$ unknowns. Positioning the stencil over each xy position where u is unknown will give the right number of equations, and the order we take the 16 points is indicated by the arrows on the diagram.



It follows that there will be a system of equations involving

$$\begin{pmatrix}
-4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \\ u_{13} \\ u_{23} \\ u_{33} \\ u_{43} \\ u_{14} \\ u_{24} \\ u_{34} \\ u_{44}
\end{pmatrix} = \ldots$$

with a right-hand side that depends on the function f and the boundary conditions.

There is a great deal of structure in this matrix. Most of the elements are zero. Apart from that there are five non-zero diagonal bands (from top-left to bottom-right), each corresponding to a component of the five-point stencil. The main diagonal is made up of repetitions of -4, the coefficient from the centre of the 5-point stencil. Immediately above and below the main diagonal are terms that come from the easterly and westerly extremes of the stencil, respectively. Separated from the tridiagonal band are two outlying lines of 1s. The uppermost sequence of 1s is due to the northerly point on the stencil and the lowermost is a consequence of the southerly point.

It is worth noting that much of this structure failed to emerge in the numerical examples considered earlier. This was because the mesh was so coarse (that is, h was so large) that the stencil was always in touch with the boundary. It is more usual that most placings of the stencil will produce an equation involving five unknowns.

In general, then, an implementation of the five-point stencil will ultimately involve having to solve a potentially large number of simultaneous equations. We have seen in HELM 30 methods for dealing with systems of equations, for example we saw the Jacobi and Gauss-Seidel iterative methods. It is possible, in the present application, to implement these methods directly *via* the numerical stencil. The next subsection describes how this may be achieved.

4. Iterative methods

An implementation of the five-point stencil

$$-4u_0 + u_E + u_W + u_S + u_N = h^2 f_0$$

leads to a system of simultaneous equations in the unknowns. This system of equations can be dealt with using methods seen in HELM 30, but here we show ways in which systematic iterative methods can be derived **directly from the numerical stencil**.
The general approach is as follows:

1. Start with an initial guess for the unknowns. Call this initial guess $u_{i,j}^0$.
2. Use some means to improve the guess. Call the improvement $u_{i,j}^1$.
3. And so on. In general we derive a new set of approximations $u_{i,j}^{n+1}$ in terms of the previous approximations $u_{i,j}^n$.

Jacobi iteration

The approach we adopt here is to update the approximation at the centre of the stencil using the four old values around the edge of the stencil. That is

$$-4u_0^{n+1} + u_E^n + u_W^n + u_S^n + u_N^n = h^2 f_0$$

rearranging this gives

$$u_0^{n+1} = \frac{1}{4}\left(u_E^n + u_W^n + u_S^n + u_N^n - h^2 f_0\right)$$

The following Example uses the same data (rounded to four decimal places here) as in Example 6.

Example 7

Suppose that $u = u(x, y)$ satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the square region $0 < x, y < 1$ with $u = x^2 y$ on the boundary. Assuming a mesh size of $h = \frac{1}{3}$ use the Jacobi iteration, with starting values $u_{ij}^0 = 0$, to perform two iterations. The boundary data are as given in the schematic below.

$y \uparrow$

0.0000	0.1111	0.4444	1.0000	
0.0000	u_{12}	u_{22}	0.6667	
0.0000	u_{11}	u_{21}	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

Solution

Putting in the initial guesses for the four unknowns $u_{11}, u_{12}, u_{21}, u_{22}$ we obtain the situation depicted below.

$y \uparrow$

0.0000	0.1111	0.4444	1.0000	
0.0000	0	0	0.6667	
0.0000	0	0	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

The first iteration involves using

$$4u_0^1 = u_E^0 + u_W^0 + u_N^0 + u_S^0 - h^2 f_0$$

where, in this case, $h^2 f_0 = 0$. So the first iteration gives us

$$u_{11}^1 = 0.0000 \qquad u_{21}^1 = 0.0833 \qquad u_{12}^1 = 0.0278 \qquad u_{22}^1 = 0.2778$$

The second iteration begins by putting these new approximations to the interior values into the grid. This gives

$y \uparrow$

0.0000	0.1111	0.4444	1.0000	
0.0000	0.0278	0.2778	0.6667	
0.0000	0.0000	0.0833	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

We now apply $4u_0^2 = u_E^1 + u_W^1 + u_N^1 + u_S^1$ to obtain

$$u_{11}^2 = 0.0278 \qquad u_{21}^2 = 0.1528 \qquad u_{12}^2 = 0.0972 \qquad u_{22}^2 = 0.3056$$

In practice, using a computer to carry out the arithmetic, we would continue iterating until the results settle down to a converged value. Using a computer spreadsheet, for example, we can see that a total of 15 iterations is enough to achieve results converged to four decimal places. We noted earlier that, to four decimal places, $u_{11} = 0.0833$, $u_{21} = 0.1944$, $u_{12} = 0.1389$ and $u_{22} = 0.3611$.

The following Task uses the same data as the preceding Task (pages 23-24), except that we have rounded the boundary data to four decimal places instead of using the exact fractions.

Task

Suppose that $u = u(x, y)$ satisfies Poisson's equation

$$u_{xx} + u_{yy} = -2$$

in the square region $0 < x, y < 1$ with $u = xy$ on the boundary. Assuming a mesh size of $h = \frac{1}{3}$ use the Jacobi iteration, with starting values $u^0_{ij} = 0$, to perform two iterations. The boundary data are as given in the schematic below.

$y \uparrow$				
0.0000	0.3333	0.6667	1.0000	
0.0000	u_{12}	u_{22}	0.6667	
0.0000	u_{11}	u_{21}	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

Your solution

First iteration:

Answer

Putting in the initial guesses for the four unknowns we obtain the situation depicted below.

$y \uparrow$				
0.0000	0.3333	0.6667	1.0000	
0.0000	0	0	0.6667	
0.0000	0	0	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

The first iteration involves using

$$4u^1_0 = u^0_E + u^0_W + u^0_N + u^0_S - h^2 f_0$$

where in this case $h^2 f_0 = -0.2222$. So the first iteration gives us

$$u^1_{11} = 0.0556 \qquad u^1_{21} = 0.1389 \qquad u^1_{12} = 0.1389 \qquad u^1_{22} = 0.3889$$

Your solution

Second iteration:

Answer

The second iteration begins by putting these new approximations to the interior values into the grid. This gives

$$
\begin{array}{lllll}
y \uparrow & & & & \\
0.0000 & 0.3333 & 0.6667 & 1.0000 & \\
0.0000 & 0.1389 & 0.3889 & 0.6667 & \\
0.0000 & 0.0556 & 0.1389 & 0.3333 & \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & \rightarrow x
\end{array}
$$

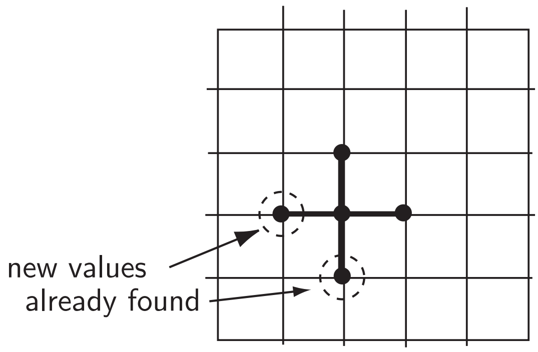
We now perform the second iteration $4u_0^2 = u_E^1 + u_W^1 + u_N^1 + u_S^1 - h^2 f_0$ again, but with the new values. We obtain

$$
\begin{aligned}
u_{11}^2 &= 0.1250 \\
u_{21}^2 &= 0.2500 \\
u_{12}^2 &= 0.2500 \\
u_{22}^2 &= 0.4583
\end{aligned}
$$

In the case above 17 iterations are required to achieve results that have converged to 4 decimal places. We find that $u_{11} = 0.2222$, $u_{12} = 0.3333$, $u_{21} = 0.3333$ and $u_{22} = 0.5556$.

Gauss-Seidel iteration

In the implementation of the Jacobi method we used old values for the southerly and westerly points when new values had already been calculated.



The Gauss-Seidel method uses the new values as soon as they are available. Stating this formally we have

$$u_0^{n+1} = \frac{1}{4}\left(u_E^n + \underbrace{u_W^{n+1} + u_S^{n+1}}_{\text{new values here}} + u_N^n - h^2 f_0\right)$$

Example 8 below uses the same data as Examples 6 and 7.

Example 8

Suppose that $u = u(x, y)$ satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the square region $0 < x, y < 1$ with $u = x^2 y$ on the boundary. Assuming a mesh size of $h = \frac{1}{3}$, use the Gauss-Seidel iteration, with starting values $u_{ij}^0 = 0$, to perform two iterations. The boundary data are as given in the schematic below.

$y \uparrow$				
0.0000	0.1111	0.4444	1.0000	
0.0000	u_{12}	u_{22}	0.6667	
0.0000	u_{11}	u_{21}	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

Solution

Putting in the initial guesses for the four unknowns we obtain the situation depicted below.

$y \uparrow$				
0.0000	0.1111	0.4444	1.0000	
0.0000	0	0	0.6667	
0.0000	0	0	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

The first iteration involves using

$$4u_0^1 = u_E^0 + u_W^1 + u_N^0 + u_S^1 - h^2 f_0$$

where in this case $h^2 f_0 = 0$. So the first iteration gives us

$$\begin{aligned} u_{11}^1 &= 0.0000 \\ u_{21}^1 &= 0.0833 \\ u_{12}^1 &= 0.0278 \\ u_{22}^1 &= 0.3056 \end{aligned}$$

Solution (contd.)

The second iteration begins by putting these new approximations to the interior values into the grid. This gives

$y \uparrow$				
0.0000	0.1111	0.4444	1.0000	
0.0000	0.0278	0.3056	0.6667	
0.0000	0.0000	0.0833	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

We now apply $4u_0^2 = u_E^1 + u_W^2 + u_N^1 + u_S^2 - h^2 f_0$ to obtain

$$u_{11}^2 = 0.0278 \qquad u_{21}^2 = 0.1667 \qquad u_{12}^2 = 0.1111 \qquad u_{22}^2 = 0.3472$$

(And, using a computer spreadsheet, for example, we can see that a total of 7 iterations is enough to achieve results converged to four decimal places. This compares well with the 15 iterations required by Jacobi in Example 7.)

Task

Suppose that $u = u(x, y)$ satisfies Poisson's equation

$$u_{xx} + u_{yy} = -2$$

in the square region $0 < x, y < 1$ with $u = xy$ on the boundary. Assuming a mesh size of $h = \frac{1}{3}$ use the Gauss-Seidel iteration, with starting values $u_{ij}^0 = 0$, to perform two iterations. The boundary data are as given in the schematic below.

$y \uparrow$				
0.0000	0.3333	0.6667	1.0000	
0.0000	u_{12}	u_{22}	0.6667	
0.0000	u_{11}	u_{21}	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

Your solution

First iteration:

Answer

Putting in the initial guesses for the four unknowns we obtain the situation depicted below.

$$\begin{array}{lllll} y\uparrow & & & & \\ 0.0000 & 0.3333 & 0.6667 & 1.0000 & \\ 0.0000 & 0 & 0 & 0.6667 & \\ 0.0000 & 0 & 0 & 0.3333 & \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & \rightarrow x \end{array}$$

The first iteration involves using

$$4u_0^1 = u_E^0 + u_W^1 + u_N^0 + u_S^1 - h^2 f_0$$

where in this case $h^2 f_0 = -0.2222$. We need to take care so as to use new values as soon as they are available So the first iteration gives us

$$\begin{aligned} u_{11}^1 &= 0.0556 && \\ u_{21}^1 &= 0.1528 && \text{using the new } u_{11} \text{ approximation} \\ u_{12}^1 &= 0.1528 && \text{using the new } u_{11} \text{ approximation} \\ u_{22}^1 &= 0.4653 && \text{using the new } u_{12} \text{ and } u_{21} \text{ approximations} \end{aligned}$$

(to 4 decimal places).

Your solution

Second iteration:

Answer

The second iteration begins by putting these new approximations to the interior values into the grid. This gives

$y \uparrow$

0.0000	0.3333	0.6667	1.0000	
0.0000	0.1528	0.4653	0.6667	
0.0000	0.0556	0.1528	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

We now apply $4u_0^2 = u_E^1 + u_W^2 + u_N^1 + u_S^2 - h^2 f_0$ again, but with the new values. We obtain

$$\begin{aligned} u_{11}^2 &= 0.1319 \\ u_{21}^2 &= 0.2882 \quad \text{using the new } u_{11} \text{ approximation} \\ u_{12}^2 &= 0.2882 \quad \text{using the new } u_{11} \text{ approximation} \\ u_{22}^2 &= 0.5330 \quad \text{using the new } u_{12} \text{ and } u_{21} \text{ approximations} \end{aligned}$$

and we can write this information in the form

$y \uparrow$

0.0000	0.3333	0.6667	1.0000	
0.0000	0.2882	0.5330	0.6667	
0.0000	0.1319	0.2882	0.3333	
0.0000	0.0000	0.0000	0.0000	$\rightarrow x$

Again, a computer can be used to continue iterating until convergence. This method applied to this Task needs 8 iterations to achieve 4 decimal place convergence, a fact which compares very well with the 17 required by the Jacobi method.

Convergence

We now summarise some important points

1. For the problems discussed in these pages, the Jacobi and Gauss-Seidel methods will **always converge for any initial guesses** u_{ij}^0. (Of course, very poor initial guesses will result in more iterations being required.)

2. For a given problem and given starting guesses u_{ij}^0, the Gauss-Seidel method will, in general, **converge in fewer iterations than Jacobi**. (That is, using the new, improved values as soon as they are available speeds up the process.)

3. One possible advantage with the Jacobi approach is that it can be **parallelised**, that is, it is in theory possible to do all the calculations for a given iteration simultaneously. In other words, everything we will need to know to carry out an iteration is known before the iteration begins. This is not the case with Gauss-Seidel in which during an iteration, most calculations use a result from within the current iteration. This advantage with Jacobi only manifests itself when using computers with a parallelisation option and for large problems.

Exercises

1. Suppose that $u = u(x, y)$ satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the square region $0 < x, y < 1$. Assuming a mesh size of $h = \frac{1}{3}$ use the Jacobi iteration, with starting values $u^0_{ij} = 0$, to perform two iterations. The boundary data are as given in the schematic below:

$y \uparrow$				
0.0000	0.2500	0.7500	1.0000	
0.4000	u_{12}	u_{22}	0.8000	
0.8000	u_{11}	u_{21}	0.4000	
0.0000	0.7500	0.2500	0.0000	$\rightarrow x$

2. Suppose that $u = u(x, y)$ satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the square region $0 < x, y < 1$. Assuming a mesh size of $h = \frac{1}{3}$ use the Gauss-Seidel iteration, with starting values $u^0_{ij} = 0$, to perform two iterations. The boundary data are as given in the schematic below.

$y \uparrow$				
0.0000	0.2500	0.7500	1.0000	
0.4000	u_{12}	u_{22}	0.8000	
0.8000	u_{11}	u_{21}	0.4000	
0.0000	0.7500	0.2500	0.0000	$\rightarrow x$

Answers

1. Putting in the initial guesses for the four unknowns we obtain the situation depicted below.

$y \uparrow$

0.0000	0.2500	0.7500	1.0000	
0.4000	0	0	0.8000	
0.8000	0	0	0.4000	
0.0000	0.7500	0.2500	0.0000	$\rightarrow x$

The first iteration involves using

$$4u_0^1 = u_E^0 + u_W^0 + u_N^0 + u_S^0 - h^2 f_0$$

where in this case $h^2 f_0 = 0.0000$. So the first iteration gives us

$$\begin{aligned} u_{11}^1 &= 0.3875 \\ u_{21}^1 &= 0.1625 \\ u_{12}^1 &= 0.1625 \\ u_{22}^1 &= 0.3875 \end{aligned}$$

The second iteration begins by putting these new approximations to the interior values into the grid. This gives

$y \uparrow$

0.0000	0.2500	0.7500	1.0000	
0.4000	0.1625	0.3875	0.8000	
0.8000	0.3875	0.1625	0.4000	
0.0000	0.7500	0.2500	0.0000	$\rightarrow x$

We now apply $4u_0^2 = u_E^1 + u_W^1 + u_N^1 + u_S^1 - h^2 f_0$ to obtain

$$\begin{aligned} u_{11}^2 &= 0.4688 \\ u_{21}^2 &= 0.3563 \\ u_{12}^2 &= 0.3563 \\ u_{22}^2 &= 0.4688 \end{aligned}$$

Answers

2. Putting in the initial guesses for the four unknowns we obtain the situation depicted below.

$y \uparrow$

0.0000	0.2500	0.7500	1.0000	
0.4000	0	0	0.8000	
0.8000	0	0	0.4000	
0.0000	0.7500	0.2500	0.0000	$\rightarrow x$

The first iteration involves using

$$4u_0^1 = u_E^0 + u_W^1 + u_N^0 + u_S^1 - h^2 f_0$$

where in this case $h^2 f_0 = 0.0000$. So the first iteration gives us

$$\begin{aligned} u_{11}^1 &= 0.3875 \\ u_{21}^1 &= 0.2594 \\ u_{12}^1 &= 0.2594 \\ u_{22}^1 &= 0.5172 \end{aligned}$$

The second iteration begins by putting these new approximations to the interior values into the grid. This gives

$y \uparrow$

0.0000	0.2500	0.7500	1.0000	
0.4000	0.2594	0.5172	0.8000	
0.8000	0.3875	0.2594	0.4000	
0.0000	0.7500	0.2500	0.0000	$\rightarrow x$

We now apply $4u_0^2 = u_E^1 + u_W^2 + u_N^1 + u_S^2 - h^2 f_0$ to obtain

$$\begin{aligned} u_{11}^2 &= 0.5172 \\ u_{21}^2 &= 0.4211 \\ u_{12}^2 &= 0.4211 \\ u_{22}^2 &= 0.5980 \end{aligned}$$