

Resisted Motion

34.3

Introduction

This Section returns to the simple models of projectiles considered in Section 34.1. It explores the magnitude of air resistance effects and the effects of including simple models of air resistance on the earlier analysis.

Prerequisites

Before starting this Section you should ...

- be able to solve second order, constant coefficient ODEs
- be able to use Newton's laws to describe and model the motion of particles

Learning Outcomes

On completion you should be able to ...

- compute the effect of air resistance proportional to velocity on particles moving under gravity
- define terminal velocity for linear and quadratic dependence of resistance on velocity

1. Resisted motion

Resistance proportional to velocity

In Section 34.2 we introduced methods of analysing the motion of projectiles on the assumption that air resistance or **drag** can be neglected. In this Section we will consider the accuracy of this assumption in some particular cases and take a look at the consequences which including air resistance has for the vector analysis of forces and motion.

Consider the subsequent motion of an object that is thrown horizontally. Let us introduce coordinate axes x (horizontal, unit vector \underline{i}) and y (vertical upwards, unit vector \underline{j}) and place the origin of coordinates at the point of release. The forces on the object consist of the weight $mg\underline{j}$ and a resisting force proportional to the velocity v . This force may be written

$$-cv = -cxi - cyj,$$

where c is a constant of proportionality. Newton's second law gives

$$m\underline{a} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) = (-c\dot{x}\underline{i} - c\dot{y}\underline{j} - mg\underline{j}).$$

This can be separated into two equations:

$$m\ddot{x} = -c\dot{x} \tag{3.1}$$

and

$$m\ddot{y} = -c\dot{y} - mg. \tag{3.2}$$

These equations each involve only one variable so they are **uncoupled**. They can be solved separately. Consider the Equation (3.1) for the horizontal motion, first in the form

$$m\ddot{x} + c\dot{x} = 0.$$

Dividing through by m and using a new constant $\kappa = c/m$,

$$\ddot{x} + \kappa\dot{x} = 0$$

A solution to this equation (see HELM 19) is

$$x = A + Be^{-\kappa t}$$

where A and B are constants. These constants may be evaluated by means of the initial conditions

$$x(0) = 0 \quad \dot{x}(0) = v_0$$

where v_0 is the speed with which the object is thrown (recall that it is thrown horizontally). The first condition gives

$$0 = A + B$$

which means that $A = -B$. The second gives

$$v_0 = -B\kappa$$

which implies that $B = -\frac{v_0}{\kappa}$, so

$$x(t) = \frac{v_0}{\kappa} (1 - e^{-\kappa t}) \tag{3.3}$$

The initial conditions for the vertical motion are

$$y(0) = 0 \quad \dot{y}(0) = 0.$$

Equation (3.2), in the form

$$\ddot{y} + \kappa\dot{y} = -g$$

may be solved by multiplying through by $e^{\kappa t}$ (HELM 19) which enables us to write

$$\frac{d}{dt} (\dot{y}e^{\kappa t}) = -ge^{\kappa t}.$$

After integrating with respect to t twice,

$$y(t) = C + De^{-\kappa t} - \frac{gt}{\kappa}.$$

The initial conditions give

$$0 = C + D \quad \text{and} \quad 0 = -\kappa D - g/\kappa$$

which means that $D = -g/\kappa^2$, so $C = g/\kappa^2$ and

$$y(t) = \frac{g}{\kappa^2}(1 - e^{-\kappa t}) - \frac{gt}{\kappa}. \quad (3.4)$$

From Equation (3.1), the horizontal component of velocity is

$$\dot{x}(t) = v_0 e^{-\kappa t}. \quad (3.5)$$

The air resistance causes the horizontal component of velocity to decrease exponentially from its original value. From Equation (3.2), the upward vertical component of velocity is

$$\dot{y}(t) = \frac{g}{\kappa} (e^{-\kappa t} - 1). \quad (3.6)$$

For very large values of t , $e^{-\kappa t}$ is near zero, so the vertical component of velocity is nearly constant at $-g/\kappa$. The negative sign indicates that the object is moving downwards. g/κ represents the terminal velocity for vertical motion under gravity for a particle subject to air resistance proportional to velocity. Sketches of the variations of the components of velocity with time are shown in Figure 30.

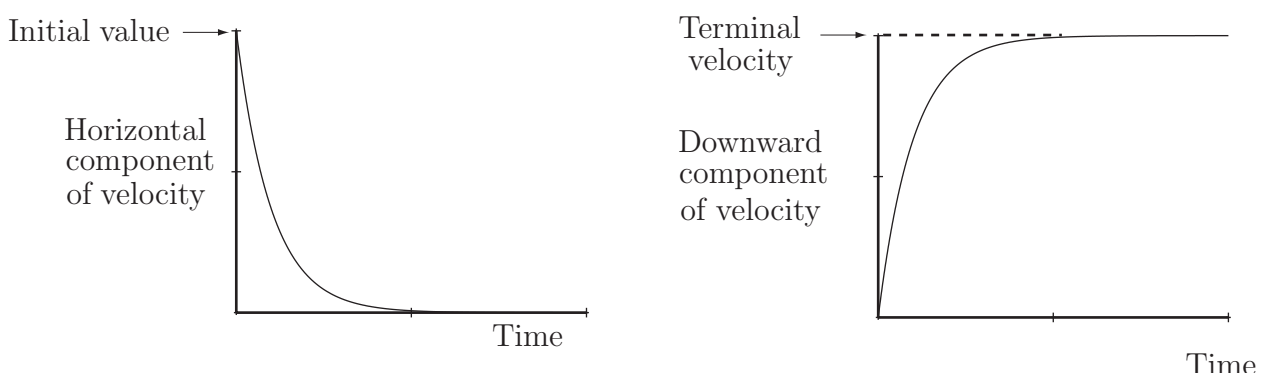


Figure 30
Velocity components of an object launched horizontally
and subject to resistance proportional to velocity

By combining the components of velocity given in (3.5) and (3.6), it is possible to obtain the magnitude and direction of the velocity of an object projected horizontally at speed v_0 and subject to air resistance proportional to velocity, the magnitude is $\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}$ and the direction is $\tan^{-1}(\dot{y}(t)/\dot{x}(t))$.

Note that the expression for terminal velocity could be obtained directly from (3.2), by setting $\ddot{y} = 0$.



Example 16

At the time that the parachute opens a parachutist of mass 100 kg is travelling horizontally at 20 m s^{-1} and is 200 m above the ground. Calculate (a) the parachutist's height above the ground and (b) the magnitude and direction of the parachutist's velocity after 10 s assuming that air resistance during the first 100 m of fall may be modelled as proportional to velocity with constant of proportionality $c = 100$.

Solution

- (a) Substituting $m = 100, g = 9.81$ and $c = 100$ in Equation (3.4) gives the distance dropped during 10 s as 88.3 m. So the parachutist will be 111.7 m above the ground after 10 s. The model is valid up to this distance.
- (b) The vertical component of velocity after 10 s is given by Equation (3.6) i.e. 9.81 m s^{-1} . The horizontal component of velocity is given by Equation (3.5) i.e. $9.08 \times 10^{-4} \text{ m s}^{-1}$, which is practically negligible. So, after 10 s, the parachutist will be moving more or less vertically downwards at 9.81 m s^{-1} .

If the object is launched at some angle θ above the horizontal, then the initial conditions on velocity are

$$\dot{x}(0) = v_0 \cos \theta \quad \dot{y}(0) = v_0 \sin \theta$$

These lead to the following equations, replacing (3.3) and (3.4):

$$x(t) = \frac{v_0 \cos \theta}{\kappa} (1 - e^{-\kappa t}) \tag{3.7}$$

$$y(t) = \left[\frac{v_0 \sin \theta}{\kappa} + \frac{g}{\kappa^2} \right] (1 - e^{-\kappa t}) - \frac{gt}{\kappa}. \tag{3.8}$$

To obtain the trajectory of the object, (3.7) can be rearranged to give

$$(1 - e^{-\kappa t}) = \frac{\kappa x}{v_0 \cos \theta} \quad \text{and} \quad t = -\frac{1}{\kappa} \ln \left(1 - \frac{\kappa x}{v_0 \cos \theta} \right).$$

These can be substituted in (3.8) to give

$$y = x \left(\tan \theta + \frac{g}{\kappa v_0 \cos \theta} \right) + \frac{g}{\kappa^2} \ln \left(1 - \frac{\kappa x}{v_0 \cos \theta} \right). \tag{3.9}$$

Figure 31 compares predictions from this result with those predicted from the result obtained by ignoring air resistance (Equations (3.1) and (3.2)). The effect of including air resistance is to change the projectile trajectory from a parabola, symmetrical about the highest point, to an asymmetric curve, resulting in reduced maximum range.

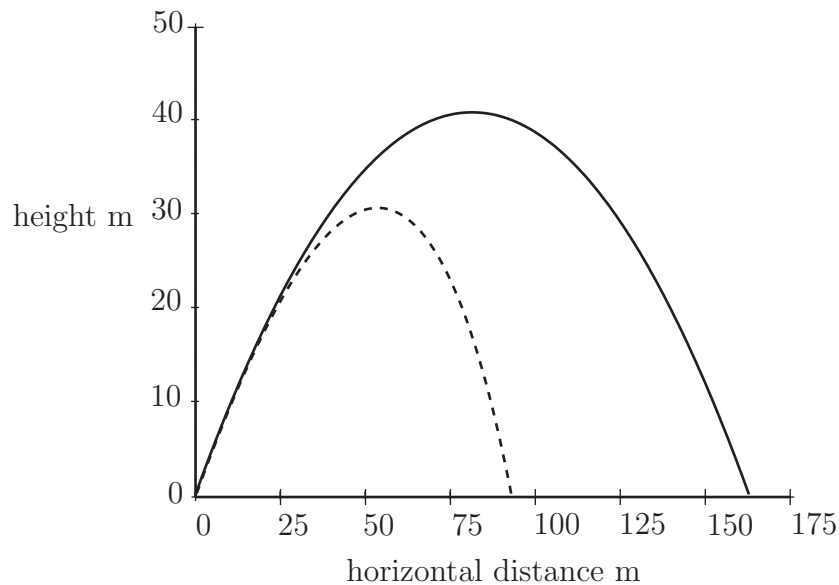


Figure 31

Predicted trajectories of an object projected at 45° with speed 40 m s^{-1} in the absence of air resistance (solid line) and with air resistance proportional to velocity such that $\kappa = 0.184$ (broken line)

Quadratic resistance

Unfortunately, it is not often very accurate to model air resistance by a force that is simply proportional to velocity. For a spherical object, a good approximation for the dependence of the air resistance force vector \underline{R} on the speed (v) and diameter (D) of the object is

$$\underline{R} = (c_1 D + c_2 D^2 |v|) \underline{v} \quad (3.10)$$

with $c_1 = 1.55 \times 10^{-4}$ and $c_2 = 0.22$ in SI units for air. As would be expected intuitively, the bigger the sphere and the faster it is moving the greater the drag it will experience. If D and $|v|$ are very small then the second term in (3.10) can be neglected compared with the first and the **linear approximation** is reasonable i.e.

$$\underline{R} \simeq c_1 D \underline{v} \quad D|v| \leq 10^{-5}. \quad (3.11)$$

Note that $c_1 \ll c_2$, so if D and $|v|$ are not very small, for example a cricket ball ($D = 0.07 \text{ m}$) moving at 40 m s^{-1} , the first term in (3.10) can be neglected compared with the second. This gives rise to the **quadratic approximation**

$$\underline{R} \simeq c_2 D^2 |v| \underline{v} \quad 10^{-2} \leq D|v| \leq 1. \quad (3.12)$$

The ranges of validity of these approximations are shown graphically in Figure 32 for a sphere of diameter 0.01 m . In general the linear approximation is accurate for small slow-moving objects and the quadratic approximation is satisfactory for larger faster objects. The linear approximation is similar to Stokes' law (first stated in 1845):

$$|\underline{R}| = 6\pi\mu r |v| \quad (3.13)$$

where μ is the coefficient of viscosity of the fluid surrounding a sphere of radius r . According to Stokes' law, $c_1 = 3\pi\mu$. This gives $c_1 = 0.17 \times 10^{-4} \text{ kg m}^{-1}\text{s}^{-1}$ for air. Similarly, the quadratic approximation is consistent with a relationship deduced by Prandtl (first stated in 1917) for a sphere:

$$|\underline{R}| = 0.625\rho r^2 |\underline{v}|^2 \quad (3.14)$$

where ρ is the density of the fluid. This implies that $c_2 = 0.625\rho/4 = 0.202 \text{ kg m}^{-3}$ for air.

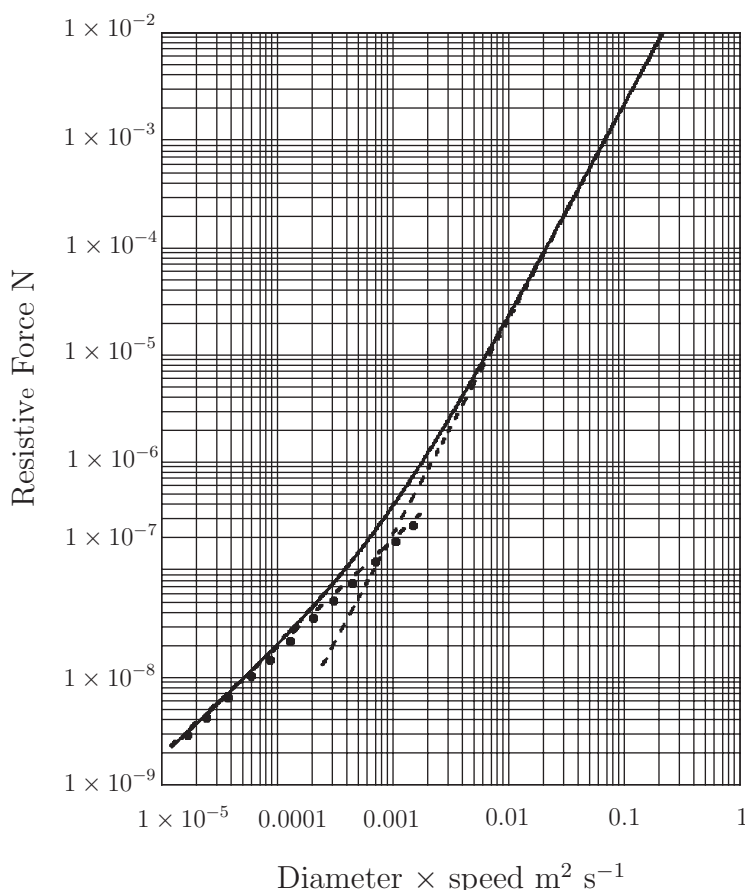


Figure 32

Resistive force, as a function of the product of diameter and speed, predicted by Equation (3.10) (solid line) and the approximations Equation (3.11) (broken line) and Equation (3.12) (dash-dot line), for a sphere of diameter 0.01m

The mathematical complexity of the equations for projectile motion in 2D resulting from the quadratic approximation is considerable. Consider an object with an initial horizontal velocity and the same coordinate axes as before, but this time the resistive force is given by $c|\underline{v}|\underline{v}$ (the quadratic approximation). For this case Newton's second law gives

$$m\underline{a} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) = -c\dot{x}\sqrt{(\dot{x}^2 + \dot{y}^2)}\underline{i} - c\dot{y}\sqrt{(\dot{x}^2 + \dot{y}^2)}\underline{j} - mg\underline{j}.$$

The corresponding scalar differential equations are

$$m\ddot{x} = -c\dot{x}\sqrt{(\dot{x}^2 + \dot{y}^2)}$$

and

$$m\ddot{y} = -c\dot{y}\sqrt{(\dot{x}^2 + \dot{y}^2)} - mg.$$

You should note that \dot{x} and \dot{y} appear in both equations and cannot be separated out. These differential equations are **coupled**. (Ways of dealing with such coupled equations is introduced in HELM 20.)

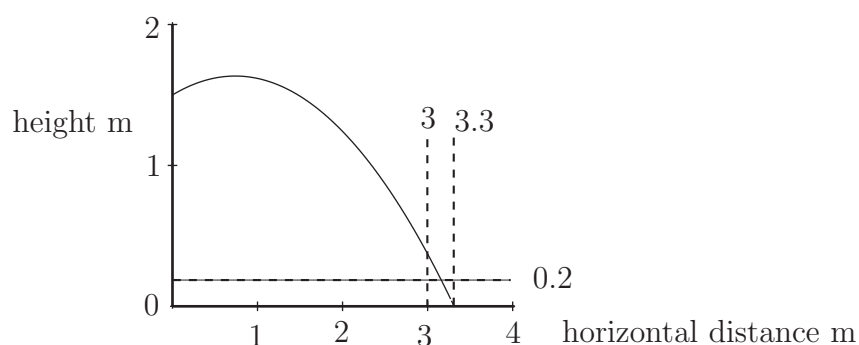


Suppose that the academic in Example 1.5 screws up sheets of paper into spheres of radius 0.03 m and mass 0.01 kg. Calculate the effect of linear air resistance on the likelihood of the chosen trajectory entering the waste paper basket.

Your solution

Answer

Since $D|v| = 4.75 \times 0.06 = 0.285$, the linear approximation for air resistance is not valid. If however it is assumed that it is, then $\kappa = c_1 D/m = 1.55 \times 10^{-4} \times 0.06/0.01 = 9.3 \times 10^{-4}$. A plot of the resulting trajectory according to Equation (3.9) is shown in the diagram below.



Predicted trajectory of paper balls with linear air resistance

With the stated assumptions, air resistance is predicted to have little or no effect on the trajectory of the paper balls.

Vertical motion with quadratic resistance

Although it is not straightforward to model motion in 2D with resistance proportional to velocity squared, it is possible to consider the motion of an object falling vertically under gravity experiencing quadratic air resistance. In this case the equation of motion may be written in terms of the (vertical) velocity ($\dot{y} = v$) as

$$m \frac{dv}{dt} = mg - cv^2.$$

This nonlinear differential equation can be solved by using separation of variables (HELM 19). First we rearrange the differential equation to give

$$\frac{dt}{dv} = \frac{-m/c}{-\frac{mg}{c} + v^2}.$$

Then we integrate both sides with respect to v , and write $\kappa_1 = c/m$ (note that this c is different from the c used for linear air resistance) which yields

$$t + C = \frac{1}{2\sqrt{g\kappa_1}} \ln \left(\frac{a + v}{a - v} \right)$$

where $a = \sqrt{g/\kappa_1}$. If the object starts from rest $C = 0$, so

$$\frac{a + v}{a - v} = e^{2t\sqrt{g\kappa_1}} \text{ and}$$

$$v = a \frac{(1 - e^{-2t\sqrt{g\kappa_1}})}{(1 + e^{-2t\sqrt{g\kappa_1}})} = a \tanh(t\sqrt{g\kappa_1}). \quad (3.15)$$

Note that for $t \rightarrow \infty$ this predicts that the terminal velocity $v_t = a = \sqrt{g/\kappa_1}$. This expression for terminal velocity may be compared with that for linear air resistance (g/κ). So the quadratic resistance model predicts a square root form for terminal velocity. Note that the expression for the terminal velocity for vertical motion of a particle subject to resistance proportional to the square of the velocity could be obtained from $m \frac{dv}{dt} = mg - cv^2$ by setting $\frac{dv}{dt} = 0$. If we write $\tau = \frac{v_t}{g}$ (note that this has units of time), then Equations (3.6) and (3.15) may be written

$$v = v_t (1 - e^{-t/\tau})$$

and

$$v = v_t \frac{(1 - e^{-2t/\tau})}{(1 + e^{-2t/\tau})}.$$

Using these expressions, it is possible to compare the variation of the ratio v/v_T as a function of time in units of τ as in Figure 33. The graph shows the intuitive result that a falling object subject to quadratic resistance approaches its terminal velocity more rapidly than a falling object subject to resistance proportional to velocity. For example, at $t/\tau = 5$, v/v_t is 0.993 with linear resistance and 0.9991 with quadratic resistance. Note however that the terminal velocities and the time steps used

in the graph are different.

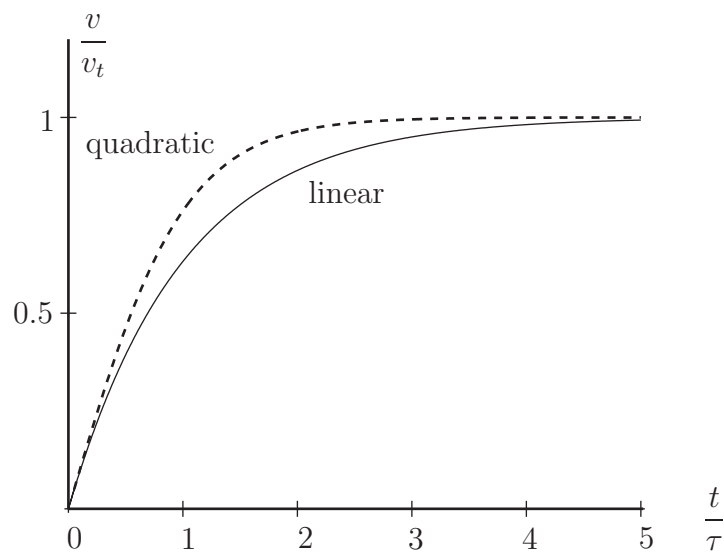


Figure 33

Comparison of the variations in vertical velocities for a falling object subject to linear and quadratic resistance



Note that the curves in Figure 33 are very close to each other and almost straight for small values of t/τ . Why should this be the case? As well as proposing an intuitive explanation, consider the result of expanding the exponential term in (3.6) in a Maclaurin power series.

Your solution

Answer

It is to be expected that, in the initial stages of motion when v and t are small, the gravitational force will dominate over air resistance i.e. $v \approx -gt$. A Maclaurin power series expansion of the exponential term in (3.6) gives

$$e^{-\kappa t} = 1 - \kappa t + \frac{1}{2}(\kappa t)^2 - \dots$$

So $e^{-\kappa t} - 1 = -\kappa t + \frac{1}{2}(\kappa t)^2 - \dots$ so $v \approx \frac{g}{\kappa} \times (-\kappa t + \frac{1}{2}(\kappa t)^2 - \dots)$

If t is much smaller than $1/\kappa$, then only the first term need be considered which gives $v \approx -gt$.