Two-Way Analysis of Variance

44.2

Introduction

In the one-way analysis of variance (Section 44.1) we consider the effect of one factor on the values taken by a variable. Very often, in engineering investigations, the effects of two or more factors are considered simultaneously.

The two-away ANOVA deals with the case where there are two factors. For example, we might compare the fuel consumptions of four car engines under three types of driving conditions (e.g. urban, rural, motorway). Sometimes we are interested in the effects of both factors. In other cases one of the factors is a 'nuisance factor' which is not of particular interest in itself but, if we allow for it in our analysis, we improve the power of our test for the other factor.

We can also allow for interaction effects between the two factors.

Prerequisites

Before starting this Section you should ...

- be familiar with the general techniques of hypothesis testing
- be familiar with the F-distribution
- be familiar with the one-way ANOVA calculations

Learning Outcomes

On completion you should be able to ...

- state the concepts and terminology of two-way ANOVA
- perform two-way ANOVA
- interpret the results of two-way ANOVA calculations

1. Two-way ANOVA without interaction

The previous Section considered a one-way classification analysis of variance, that is we looked at the variations induced by one set of values of a factor (or treatments as we called them) by partitioning the variation in the data into components representing 'between treatments' and 'within treatments.'

In this Section we will look at the analysis of variance involving two factors or, as we might say, two sets of treatments. In general terms, if we have two factors say A and B, there is no absolute reason to assume that there is no interaction between the factors. However, as an introduction to the two-way analysis of variance, we will consider the case occurring when there is no interaction between factors and an experiment is run only once. Note that some authors take the view that interaction may occur and that the residual sum of squares contains the effects of this interaction even though the analysis does not, at this stage, allow us to separate it out and check its possible effects on the experiment.

The following example builds on the previous example where we looked at the one-way analysis of variance.

Example of variance in data

In Section 44.1 we considered an example concerning four machines producing alloy spaces. This time we introduce an extra factor by considering both the machines producing the spacers and the performance of the operators working with the machines. In this experiment, the data appear as follows (spacer lengths in mm). Each operator made one spacer with each machine.

Operator	**Machine 1**	**Machine 2**	**Machine 3**	**Machine 4**
1	46	56	55	47
2	54	55	51	56
3	48	56	50	58
4	46	60	51	59
5	51	53	53	55

In a case such as this we are looking for discernible difference between the operators ('operator effects') on the one hand and the machines ('machine effects') on the other.

We suppose that the observation for operator i and machine j is taken from a normal distribution with mean

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

Here α_i is an operator effect and β_j is a machine effect. Our hypotheses may be stated as follows.

$$\text{Operator Effects} \begin{cases} H_0: \ \mu_{1j} = \mu_{2j} = \mu_{3j} = \mu_{4j} = \mu_{5j} = \mu + \beta_j \\ \qquad \text{That is } \ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0 \\ H_1: \ \text{At least one of the operator effects is different to the others} \end{cases}$$

$$\text{Machine Effects} \begin{cases} H_0: \ \mu_{i1} = \mu_{i2} = \mu_{i3} = \mu_{i4} = \mu + \alpha_i \\ \qquad \text{That is } \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0 \\ H_1: \ \text{At least one of the machine effects is different to the others} \end{cases}$$

Note that the five operators and four machines give rise to data which has only one observation per 'cell.' For example, operator 2 using machine 3 produces a spacer 51 mm long, while operator 1 using machine 2 produces a spacer which is 56 mm long. Note also that in this example we have referred to the machines by number and not by letter. This is not particularly important but it will simplify some of the notation used when we come to write out a general two-way ANOVA table shortly. We obtain one observation per cell and cannot measure variation within a cell. In this case we cannot check for interaction between the operator and the machine - the two factors used in this example. Running an experiment several times results in multiple observations per cell and in this case we should assume that there may be interaction between the factors and check for this. In the case considered here (no interaction between factors), the required sums of squares build easily on the relationship used in the one-way analysis of variance

$$SS_T = SS_{Tr} + SS_E$$

to become

$$SS_T = SS_A + SS_B + SS_E$$

where SS_A represent the sums of squares corresponding to factors A and B. In order to calculate the required sums of squares we lay out the table slightly more efficiently as follows.

Operator (j)	**Machine** (i)				**Operator Means** ($\bar{X}_{.j}$)	$(\bar{X}_{.j} - \bar{\bar{X}})$	**Operator** SS $(\bar{X}_{.j} - \bar{\bar{X}})^2$
	1	2	3	4			
1	46	56	55	47	51	-2	4
2	54	55	51	56	54	1	1
3	48	56	50	58	53	0	0
4	46	60	51	59	54	1	1
5	51	53	53	55	53	0	0
Machine Means ($\bar{X}_{i.}$)	49	56	52	55	$\bar{\bar{X}} = 53$	Sum $= 0$	$6 \times 4 = 24$

$(\bar{X}_{.j} - \bar{\bar{X}})$				
-4	3	-1	2	Sum $= 0$
Machine SS $(\bar{X}_{.j} - \bar{\bar{X}})^2$				
16	9	1	4	$30 \times 5 = 150$

Note 1

The . notation means that summation takes place over that variable. For example, the five operator means $\bar{X}_{.j}$ are obtained as $\bar{X}_{.1} = \dfrac{46+56+55+47}{4} = 51$ and so on, while the four machine means $\bar{X}_{i.}$ are obtained as $\bar{X}_{1.} = \dfrac{46+54+48+46+51}{5} = 49$ and so on. Put more generally (and this is just an example)

$$\bar{X}_{.j} = \frac{\displaystyle\sum_{i=1}^{m} x_{ij}}{m}$$

Note 2

Multiplying factors were used in the calculation of the machine sum of squares (four in this case since there are four machines) and the operator sum of squares (five in this case since there are five operators).

Note 3

The two statements 'Sum = 0' are included purely as arithmetic checks.

We also know that $SS_O = 24$ and $SS_M = 150$.

Calculating the error sum of squares

Note that the total sum of squares is easy to obtain and that the error sum of squares is then obtained by straightforward subtraction.

The total sum of squares is given by summing the quantities $(X_{ij} - \bar{\bar{X}})^2$ for the table of entries. Subtracting $\bar{\bar{X}} = 53$ from each table member and squaring gives:

Operator (j)	**Machine** (i)			
	1	2	3	4
1	49	9	4	36
2	1	4	4	9
3	25	9	9	25
4	49	49	4	36
5	4	0	0	4

The total sum of squares is $SS_T = 330$.

The error sum of squares is given by the result

$$\begin{aligned} SS_E &= SS_T - SS_A - SS_B \\ &= 330 - 24 - 150 \\ &= 156 \end{aligned}$$

At this stage we display the general two-way ANOVA table and then particularise the table for the example we are engaged in and draw conclusions by using the test as we have previously done with one-way ANOVA.

A General Two-Way ANOVA Table

Source of Variation	Sum of Squares SS	Degrees of Freedom	Mean Square MS	Value of F Ratio
Between samples (*due to factor A*) Differences between means $\bar{X}_{i.}$ and $\bar{\bar{X}}$	$SS_A = b\sum_{i=1}^{a}\left(\bar{X}_{i.}-\bar{\bar{X}}\right)^2$	$(a-1)$	$MS_A = \frac{SS_A}{(a-1)}$	$F = \frac{MS_A}{MS_E}$
Between samples (*due to factor B*) Differences between means $\bar{X}_{.j}$ and $\bar{\bar{X}}$	$SS_B = a\sum_{j=1}^{b}\left(\bar{X}_{.j}-\bar{\bar{X}}\right)^2$	$(b-1)$	$MS_B = \frac{SS_B}{(b-1)}$	$F = \frac{MS_B}{MS_E}$
Within samples (*due to chance errors*) Differences between individual observations and fitted values.	$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i.}-\bar{X}_{.j}+\bar{\bar{X}}\right)^2$	$(a-1)\times(b-1)$	$MS_E = \frac{SS_E}{(a-1)(b-1)}$	
Totals	$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{\bar{X}}\right)^2$	$(ab-1)$		

Hence the two-way ANOVA table for the example under consideration is

Source of Variation	Sum of Squares SS	Degrees of Freedom	Mean Square MS	Value of F Ratio
Between samples (*due to factor A*) Differences between means $\bar{X}_{i.}$ and $\bar{\bar{X}}$	24	4	$\frac{24}{4} = 6$	$F = \frac{6}{13} = 0.46$
Between samples (*due to factor B*) Differences between means $\bar{X}_{.j}$ and $\bar{\bar{X}}$ height	150	3	$\frac{150}{3} = 50$	$F = \frac{50}{13} = 3.85$
Within samples (*due to chance errors*) Differences between individual observations and fitted values.	156	12	$\frac{156}{12} = 13$	
TOTALS	330	19		

From the F-tables (at the end of the Workbook) $F_{4,12} = 3.26$ and $F_{3,12} = 3.49$. Since $0.46 < 3.26$ we conclude that we **do not** have sufficient evidence to reject the null hypothesis that there is no difference between the operators. Since $3.85 > 3.49$ we conclude that we **do** have sufficient evidence at the 5% level of significance to reject the null hypothesis that there in no difference between the machines.

Key Point 2

If we have two factors, A and B, with a levels of factor A and b levels of factor B, and one observation per cell, we can calculate the sum of squares as follows.

The sum of squares for factor A is

$$SS_A = \frac{1}{b}\sum_{i=1}^{a} A_i^2 - \frac{G^2}{N} \qquad \text{with } a-1 \text{ degrees of freedom}$$

and the sum of squares for factor B is

$$SS_B = \frac{1}{a}\sum_{j=1}^{b} B_j^2 - \frac{G^2}{N} \qquad \text{with } b-1 \text{ degrees of freedom}$$

where

$$A_i = \sum_{j=1}^{b} X_{ij} \text{ is the total for level } i \text{ of factor } A,$$

$$B_j = \sum_{i=1}^{a} X_{ij} \text{ is the total for level } j \text{ of factor } B,$$

$$G = \sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij} \text{ is the overall total of the data, and}$$

$N = ab$ is the total number of observations.

The total sum of squares is

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij}^2 - \frac{G^2}{N} \qquad \text{with } N-1 \text{ degrees of freedom}$$

The within-samples, or 'error', sum of squares can be found by subtraction. So

$$SS_E = SS_T - SS_A - SS_B$$

with

$$\begin{aligned}(N-1)-(a-1)-(b-1) &= (ab-1)-(a-1)-(b-1)\\ &= (a-1)(b-1) \text{ degrees of freedom}\end{aligned}$$

Task

A vehicle manufacturer wishes to test the ability of three types of steel-alloy panels to resist corrosion when three different paint types are applied. Three panels with differing steel-alloy composition are coated with three types of paint. The following coded data represent the ability of the painted panels to resist weathering.

Paint Type	Steel-Alloy 1	Steel-Alloy 2	Steel-Alloy 3
1	40	51	56
2	54	55	50
3	47	56	50

Use a two-way ANOVA procedure to determine whether any difference in the ability of the panels to resist corrosion may be assigned to either the type of paint or the steel-alloy composition of the panels.

Your solution

Do your working on separate paper and enter the main conclusions here.

Answer

Our hypotheses may be stated as follows.

Paint type $\begin{cases} H_0 : \mu_1 = \mu_2 = \mu_3 \\ H_1 : \text{ At least one of the means is different from the others} \end{cases}$

Steel-Alloy $\begin{cases} H_0 : \mu_1 = \mu_2 = \mu_3 \\ H_1 : \text{ At least one of the means is different from the others} \end{cases}$

Following the methods of calculation outlined above we obtain:

Paint Type (j)	**Steel-Alloy** (i)			**Paint Means** ($\bar{X}_{.j}$)	$(\bar{X}_{.j} - \bar{\bar{X}})$	**Paint** SS $(\bar{X}_{.j} - \bar{\bar{X}})^2$
	1	2	3			
1	40	51	56	49	-2	4
2	54	55	50	53	2	4
3	47	54	52	51	0	0
Steel-Alloy Means ($\bar{X}_{i.}$)	47	54	52	$\bar{\bar{X}} = 51$	Sum $= 0$	$8 \times 3 = 24$
	$(\bar{X}_{.j} - \bar{\bar{X}})$					
	-4	3	1	Sum $= 0$		
	Steel-Alloy SS $(\bar{X}_{.j} - \bar{\bar{X}})^2$					
	16	9	1	$26 \times 3 = 78$		

Hence $SS_{Pa} = 24$ and $SS_{St} = 78$. We now require SS_E. The calculations are as follows.

In the table below, the predicted outputs are given in parentheses.

Paint Type (j)	**Machine** (i)			**Paint Means** ($\bar{X}_{.j}$)	$(\bar{X}_{.j} - \bar{\bar{X}})$
	1	2	3		
1	40 (45)	51 (52)	56 (50)	49	-2
2	54 (49)	55 (56)	50 (54)	53	2
3	47 (47)	56 (54)	50 (52)	51	0
Steel-Alloy Means ($\bar{X}_{i.}$)	47	54	52	$\bar{\bar{X}} = 51$	Sum $= 0$
	$(\bar{X}_{.j} - \bar{\bar{X}})$				
	-4	3	1	Sum $= 0$	

Answers continued

A table of squared residuals is easily obtained as

Paint (j)	**Steel** (i)		
	1	2	3
1	25	1	36
2	25	1	16
3	0	4	4

Hence the residual sum of squares is $SS_E = 112$. The total sum of squares is given by subtracting $\bar{\bar{X}} = 51$ from each table member and squaring to obtain

Paint (j)	**Steel** (i)		
	1	2	3
1	121	0	25
2	9	16	1
3	16	25	1

The total sum of squares is $SS_T = 214$. We should now check to see that $SS_T = SS_{Pa} + SS_{St} + SS_E$. Substitution gives $214 = 24 + 78 + 112$ which is correct.

The values of F are calculated as shown in the ANOVA table below.

Source of Variation	Sum of Squares SS	Degrees of Freedom	Mean Square MS	Value of F Ratio
Between samples (due to treatment A, say, paint)	24	2	$MS_A = \frac{24}{12} = 12$	$F = \frac{12}{28} = 0.429$
Between samples (due to treatment B, say, Steel − Alloy)	78	2	$MS_B = \frac{78}{2} = 39$	$F = \frac{39}{28} = 1.393$
Within samples (due to chance errors)	112	4	$MS_E = \frac{112}{4} = 28$	
Totals	214	8		

From the F-tables the critical values of $F_{2,4} = 6.94$ and since both of the calculated F values are less than 6.94 we conclude that we do not have sufficient evidence to reject either null hypothesis.

2. Two-way ANOVA with interaction

The previous subsection looked at two-way ANOVA under the assumption that there was no interaction between the factors A and B. We will now look at the developments of two-way ANOVA to take into account possible interaction between the factors under consideration. The following analysis allows us to test to see whether we have sufficient evidence to reject the null hypothesis that the amount of interaction is effectively zero.

To see how we might consider interaction between factors A and B taking place, look at the following table which represents observations involving a two-factor experiment.

	Factor B				
Factor A	**1**	**2**	**3**	**4**	**5**
1	3	5	1	9	12
2	4	6	2	10	13
3	6	8	4	12	15

A brief inspection of the numbers in the five columns reveals that there is a constant difference between any two rows as we move from column to column. Similarly there is a constant difference between any two columns as we move from row to row. While the data are clearly contrived, it does illustrate that in this case that no interaction arises from variations in the differences between either rows or columns. Real data do not exhibit such behaviour in general of course, and we expect differences to occur and so we must check to see if the differences are large enough to provide sufficient evidence to reject the null hypothesis that the amount of interaction is effectively zero.

Notation

Let a represent the number of 'levels' present for factor A, denoted $i = 1, \ldots, a$.

Let b represent the number of 'levels' present for factor B, denoted $j = 1, \ldots, b$.

Let n represent the number of observations per cell. We assume that it is the same for each cell.

In the table above, $a = 3,\ b = 5, n = 1$. In the examples we shall consider, n will be greater than 1 and we will be able to check for interaction between the factors.

We suppose that the observations at level i of factor A and level j of factor B are taken from a normal distribution with mean μ_{ij}. When we assumed that there was no interaction, we used the **additive model**

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

So, for example, the difference $\mu_{i1} - \mu_{i2}$ between the means at levels 1 and 2 of factor B is equal to $\beta_1 - \beta_2$ and does not depend upon the level of factor A. When we allow interaction, this is not necessarily true and we write

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

Here γ_{ij} is an **interaction effect**. Now $\mu_{i1} - \mu_{i2} = \beta_1 - \beta_2 + \gamma_{i1} - \gamma_{i2}$ so the difference between two levels of factor B depends on the level of factor A.

Fixed and random effects

Often the levels assigned to a factor will be chosen deliberately. In this case the factors are said to be *fixed* and we have a **fixed effects model**. If the levels are chosen at random from a population of all possible levels, the factors are said to be *random* and we have a **random effects model**. Sometimes one factor may be *fixed* while one may be *random*. In this case we have a **mixed effects model**. In effect, we are asking whether we are interested in certain particular levels of a factor (fixed effects) or whether we just regard the levels as a sample and are interested in the population in general (random effects).

Calculation method

The data you will be working with will be set out in a manner similar to that shown below.

The table assumes n observations per cell and is shown along with a variety of totals and means which will be used in the calculations of the various test statistics to follow.

	Factor B						
Factor A	**Level 1**	**Level 2**	$\ldots$	**Level j**	$\ldots$	**Level b**	**Totals**
Level 1	x_{111} $\vdots$ x_{11n}	x_{121} $\vdots$ x_{12n}	$\ldots$	x_{1j1} $\vdots$ x_{1jn}	$\ldots$	x_{1b1} $\vdots$ x_{1bn}	$T_{1\cdot\cdot}$
Level 2	x_{211} $\vdots$ x_{21n}	x_{221} $\vdots$ x_{22n}	$\ldots$	x_{2j1} $\vdots$ x_{2jn}	$\ldots$	x_{2b1} $\vdots$ x_{2bn}	$T_{2\cdot\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
Level i	x_{i11} $\vdots$ x_{i1n}	Sum of data in cell (i,j) is $T_{ij\cdot} = \sum_{k=1}^{n} x_{ijk}$		x_{ij1} $\vdots$ x_{ijn}	$\ldots$	x_{ib1} $\vdots$ x_{ibn}	$T_{i\cdot\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
Level a	x_{a11} $\vdots$ x_{a1n}	x_{a21} $\vdots$ x_{a2n}	$\ldots$	x_{aj1} $\vdots$ x_{ajn}	$\ldots$	x_{ab1} $\vdots$ x_{abn}	$T_{a\cdot\cdot}$
Totals	$T_{\cdot 1\cdot}$	$T_{\cdot 2\cdot}$	$\ldots$	$T_{\cdot j\cdot}$	$\ldots$	$T_{\cdot b\cdot}$	$T_{\cdots}$

Notes

(a) $T_{\cdots}$ represents the grand total of the data values so that

$$T_{\cdots} = \sum_{j=1}^{b} T_{\cdot j\cdot} = \sum_{i=1}^{a} T_{i\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} x_{ijk}$$

(b) $T_{i\cdot\cdot}$ represents the total of the data in the ith row.

(c) $T_{\cdot j\cdot}$ represents the total of the data in the jth column.

(d) The total number of data entries is given by $N = nab$.

Partitioning the variation

We are now in a position to consider the partition of the total sum of the squared deviations from the overall mean which we estimate as

$$\bar{\bar{\bar{x}}} = \frac{T_{...}}{N}$$

The total sum of the squared deviations is

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(x_{ijk} - \bar{\bar{\bar{x}}}\,)^2$$

and it can be shown that this quantity can be written as

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

where SS_T is the total sum of squares given by

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} x_{ijk}^2 - \frac{T_{...}^2}{N};$$

SS_A is the sum of squares due to variations caused by factor A given by

$$SS_A = \sum_{i=1}^{a}\frac{T_{i..}^2}{bn} - \frac{T_{...}^2}{N}$$

SS_B is the sum of squares due to variations caused by factor B given by

$$SS_B = \sum_{j=1}^{b}\frac{T_{.j.}^2}{an} - \frac{T_{...}^2}{N}$$

Note that bn means $b \times n$ which is the number of observations at each level of A and an means $a \times n$ which is the number of observations at each level of B.

SS_{AB} is the sum of the squares due to variations caused by the interaction of factors A and B and is given by

$$SS_{AB} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{T_{ij.}^2}{n} - \frac{T_{...}^2}{N} - SS_A - SS_B.$$

Note that the quantity $T_{ij.} = \sum_{k=1}^{n} x_{ijk}$ is the sum of the data in the $(i,j)^{\text{th}}$ cell and that the quantity $\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{T_{ij.}^2}{n} - \frac{T_{...}^2}{N}$ is the sum of the squares between cells.

SS_E is the sum of the squares due to chance or experimental error and is given by

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}$$

The number of degrees of freedom $(N-1)$ is partitioned as follows:

SS_T	SS_A	SS_B	SS_{AB}	SS_E
$N-1$	$a-1$	$b-1$	$(a-1)(b-1)$	$N-ab$

Note that there are $ab - 1$ degrees of freedom between cells and that the number of degrees of freedom for SS_{AB} is given by

$$ab - 1 - (a-1) - (b-1) = (a-1)(b-1)$$

This gives rise to the following two-way ANOVA tables.

Two-Way ANOVA Table - Fixed-Effects Model

Source of Variation	Sum of squares SS	Degrees of Freedom	Mean Square MS	Value of F Ratio
Factor A	SS_A	$(a-1)$	$MS_A = \dfrac{SS_A}{(a-1)}$	$F = \dfrac{MS_A}{MS_E}$
Factor B	SS_B	$(b-1)$	$MS_B = \dfrac{SS_B}{(b-1)}$	$F = \dfrac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1) \times (b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F = \dfrac{MS_{AB}}{MS_E}$
Residual Error	SS_E	$(N-ab)$	$MS_E = \dfrac{SS_E}{N-ab}$	
Totals	SS_T	$(N-1)$		

Two-Way ANOVA Table - Random-Effects Model

Source of Variation	Sum of squares SS	Degrees of Freedom	Mean Square MS	Value of F Ratio
Factor A	SS_A	$(a-1)$	$MS_A = \dfrac{SS_A}{(a-1)}$	$F = \dfrac{MS_A}{MS_{AB}}$
Factor B	SS_B	$(b-1)$	$MS_B = \dfrac{SS_B}{(b-1)}$	$F = \dfrac{MS_B}{MS_{AB}}$
Interaction	SS_{AB}	$(a-1) \times (b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F = \dfrac{MS_{AB}}{MS_E}$
Residual Error	SS_E	$(N-ab)$	$MS_E = \dfrac{SS_E}{N-ab}$	
Totals	SS_T	$(N-1)$		

Two-Way ANOVA Table - Mixed-Effects Model

Case (i) A fixed and B random.

Source of Variation	**Sum of squares SS**	**Degrees of Freedom**	**Mean Square MS**	**Value of F Ratio**
Factor A	SS_A	$(a-1)$	$MS_A = \dfrac{SS_A}{(a-1)}$	$F = \dfrac{MS_A}{MS_{AB}}$
Factor B	SS_B	$(b-1)$	$MS_B = \dfrac{SS_B}{(b-1)}$	$F = \dfrac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)\times(b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F = \dfrac{MS_{AB}}{MS_E}$
Residual Error	SS_E	$(N-ab)$	$MS_E = \dfrac{SS_E}{N-ab}$	
Totals	SS_T	$(N-1)$		

Case (ii) A random and B fixed.

Source of Variation	**Sum of squares SS**	**Degrees of Freedom**	**Mean Square MS**	**Value of F Ratio**
Factor A	SS_A	$(a-1)$	$MS_A = \dfrac{SS_A}{(a-1)}$	$F = \dfrac{MS_A}{MS_E}$
Factor B	SS_B	$(b-1)$	$MS_B = \dfrac{SS_B}{(b-1)}$	$F = \dfrac{MS_B}{MS_{AB}}$
Interaction	SS_{AB}	$(a-1)\times(b-1)$	$MS_{AB} = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$F = \dfrac{MS_{AB}}{MS_E}$
Residual Error	SS_E	$(N-ab)$	$MS_E = \dfrac{SS_E}{N-ab}$	
Totals	SS_T	$(N-1)$		

Example 1

In an experiment to compare the effects of weathering on paint of three different types, two identical surfaces coated with each type of paint were exposed in each of four environments. Measurements of the degree of deterioration were made as follows.

	Environment 1		**Environment 2**		**Environment 3**		**Environment 4**	
Paint A	10.89	10.74	9.94	11.25	9.88	10.13	14.11	12.84
Paint B	12.28	13.11	14.45	11.17	11.29	11.10	13.44	11.37
Paint C	10.68	10.30	10.89	10.97	10.61	11.00	12.22	11.32

Making the assumptions of normality, independence and equal variance, derive the appropriate ANOVA tables and state the conclusions which may be drawn at the 5% level of significance in the following cases.

(a) The types of paint and the environments are chosen deliberately because the interest is in these paints and these environments.

(b) The types of paint are chosen deliberately because the interest is in these paints but the environments are regarded as a sample of possible environments.

(c) The types of paint are regarded as a random sample of possible paints and the environments are regarded as a sample of possible environments.

Solution

We know that case (a) is described as a fixed-effects model, case (b) is described as a mixed-effects model (paint type fixed) and case (c) is described as a random-effects model. In all three cases the calculations necessary to find MS_P (paints), MS_N (environments), MS_P and MS_N are identical. Only the calculation and interpretation of the test statistics will be different. The calculations are shown below.

Subtracting 10 from each observation, the data become:

	Environment 1		**Environment 2**		**Environment 3**		**Environment 4**		**Total**
Paint A	0.89	0.74	−0.06	1.25	−0.12	0.13	4.11	2.84	9.78
	(total 1.63)		(total 1.19)		(total 0.01)		(total 6.95)		
Paint B	2.28	3.11	4.45	1.17	1.29	1.10	3.44	1.37	18.21
	(total 5.39)		(total 5.62)		(total 2.39)		(total 4.81)		
Paint C	0.68	0.30	0.89	0.97	0.61	1.00	2.22	1.32	7.99
	(total 0.98)		(total 1.86)		(total 1.61)		(total 3.54)		
Total	8.00		8.67		4.01		15.30		35.98

The total sum of squares is

$$SS_T = 0.89^2 + 0.74^2 + \ldots + 1.32^2 - \frac{35.98^2}{24} = 36.910$$

We can simplify the calculation by finding the between samples sum of squares

$$SS_S = \frac{1}{2}(1.63^2 + 5.39^2 + \ldots + 3.54^2) - \frac{35.98^2}{24} = 26.762$$

Solution (contd.)

Sum of squares for paints is

$$SS_P = \frac{1}{8}(9.78^2 + 18.15^2 + 7.99^2) - \frac{35.98^2}{24} = 7.447$$

Sum of squares for environments is

$$SS_N = \frac{1}{6}(8.00^2 + 8.67^2 + 3.98^2 + 15.30^2) - \frac{35.98^2}{24} = 10.950$$

So the interaction sum of squares is $SS_{PN} = SS_S - SS_P - SS_N = 8.365$ and

the residual sum of squares is $SS_E = SS_T - SS_S = 10.148$ The results are combined in the following ANOVA table

	Deg. of Freedom	Sum of Squares	Mean Square	Variance Ratio (fixed)	Variance Ratio (mixed)	Variance Ratio (random)
Paints	2	7.447	3.724	4.40	2.67	2.67
				$F_{2,12} = 3.89$	$F_{2,6} = 5.14$	$F_{2,6} = 5.14$
Environments	3	10.950	3.650	4.31	4.31	2.61
				$F_{3,12} = 3.49$	$F_{3,12} = 3.49$	$F_{3,6} = 4.76$
Interaction	6	8.365	1.394	1.65	1.65	1.65
				$F_{6,12} = 3.00$	$F_{6,12} = 3.00$	$F_{6,12} = 3.00$
Treatment combinations	11	26.762	2.433			
Residual	12	10.148	0.846			
Total	23	36.910				

The following conclusions may be drawn. There is insufficient evidence to support the interaction hypothesis in any case. Therefore we can look at the tests for the main effects.

Case (a) Since $4.40 > 3.89$ we have sufficient evidence to conclude that paint type affects the degree of deterioration. Since $4.07 > 3.49$ we have sufficient evidence to conclude that environment affects the degree of deterioration.

Case (b) Since $2.67 < 5.14$ we do not have sufficient evidence to reject the hypothesis that paint type has no effect on the degree of deterioration. Since $4.07 > 3.49$ we have sufficient evidence to conclude that environment affects the degree of deterioration.

Case (c) Since $2.67 < 5.14$ we do not have sufficient evidence to reject the hypothesis that paint type has no effect on the degree of deterioration. Since $2.61 < 4.76$ we do not have sufficient evidence to reject the hypothesis that environment has no effect on the degree of deterioration.

If the test for interaction had given a significant result then we would have concluded that there was an interaction effect. Therefore the differences between the average degree of deterioration for different paint types would have depended on the environment and there might have been no overall 'best paint type'. We would have needed to compare combinations of paint types and environments. However the relative sizes of the mean squares would have helped to indicate which effects were most important.

Task

A motor company wishes to check the influences of tyre type and shock absorber settings on the roadholding of one of its cars. Two types of tyre are selected from the tyre manufacturer who normally provides tyres for the company's new vehicles. A shock absorber with three possible settings is chosen from a range of shock absorbers deemed to be suitable for the car. An experiment is conducted by conducting roadholding tests using each tyre type and shock absorber setting. The (coded) data resulting from the experiment are given below.

Factor	**Shock Absorber Setting**		
Tyre	**B1=Comfort**	**B2=Normal**	**B3=Sport**
Type A1	5	8	6
	6	5	9
	8	3	12
Type A2	9	10	12
	7	9	10
	7	8	9

Decide whether an appropriate model has random-effects, mixed-effects or fixed-effects and derive the appropriate ANOVA table. State clearly any conclusions that may be drawn at the 5% level of significance.

Your solution

Do the calculations on separate paper and use the space here and on the following page for your summary and conclusions.

Answer

We know that both the tyres and the shock absorbers are not chosen at random from populations consisting of all possible tyre types and shock absorber types so that their influence is described by a fixed-effects model. The calculations necessary to find MS_A, MS_B, MS_{AB} and MS_E are shown below.

	B1	**B2**	**B3**	**Totals**
A1	5 6 8	8 5 3	6 9 12	
	$T_{11} = 19$	$T_{12} = 16$	$T_{13} = 27$	$T_{1..} = 62$
A2	9 7 7	10 9 8	12 10 9	
	$T_{21} = 23$	$T_{22} = 27$	$T_{23} = 31$	$T_{2..} = 81$
Totals	$T_{.1.} = 42$	$T_{.2.} = 43$	$T_{.3.} = 58$	$T_{...} = 143$

The sums of squares calculations are:

$$SS_T = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk}^2 - \frac{T_{...}^2}{N} = 5^2 + 6^2 + \ldots + 10^2 + 9^2 - \frac{143^2}{18} = 1233 - \frac{143^2}{18} = 96.944$$

$$SS_A = \sum_{i=1}^{2} \frac{T_{i..}^2}{bn} - \frac{T_{...}^2}{N} = \frac{62^2 + 81^2}{3 \times 3} - \frac{143^2}{18} = \frac{10405}{9} - \frac{143^2}{18} = 20.056$$

$$SS_B = \sum_{j=1}^{3} \frac{T_{.j.}^2}{an} - \frac{T_{...}^2}{N} = \frac{42^2 + 43^2 + 58^2}{2 \times 3} - \frac{143^2}{18} = \frac{6977}{6} - \frac{143^2}{18} = 26.778$$

$$SS_{AB} = \sum_{i=1}^{2}\sum_{j=1}^{3} \frac{T_{ij.}^2}{n} - \frac{T_{...}^2}{N} - SS_A - SS_B = \frac{19^2 + \ldots + 31^2}{3} - \frac{143^2}{18} - 20.056 - 26.778$$

$$= \frac{3565}{3} - \frac{143^2}{18} - 20.056 - 26.778 = 5.444$$

$$SS_E = SS_T - SS_A - SS_B - SS_{AB} = 96.944 - 20.056 - 26.778 - 5.444 = 44.666$$

The results are combined in the following ANOVA table.

Source	**SS**	**DoF**	MS	F **(Fixed)**	F **(Fixed)**
Factor A	20.056	1	20.056	$\frac{MS_A}{MS_E}$	5.39 $F_{1,12} = 4.75$
Factor B	26.778	2	13.389	$\frac{MS_B}{MS_E}$	3.60 $F_{2,12} = 3.89$
Interaction AB	5.444	2	2.722	$\frac{MS_{AB}}{MS_E}$	0.731 $F_{2,12} = 3.89$
Residual E	44.666	12	3.722		
Totals	96.944	17			

Answer

The following conclusions may be drawn:

Interaction: There is insufficient evidence to support the hypothesis that interaction takes place between the factors.

Factor A: Since $5.39 > 4.75$ we have sufficient evidence to reject the hypothesis that tyre type does not affect the roadholding of the car.

Factor B: Since $3.60 < 3.89$ we do not have sufficient evidence to reject the hypothesis that shock absorber settings do not affect the roadholding of the car.

Task

The variability of a measured characteristic of an electronic assembly is a source of trouble for a manufacturer with global manufacturing and sales facilities. To investigate the possible influences of assembly machines and testing stations on the characteristic, an engineer chooses three testing stations and three assembly machines from the large number of stations and machines in the possession of the company. For each testing station - assembly machine combination, three observations of the characteristic are made.

The (coded) data resulting from the experiment are given below.

Factor	**Testing Station**		
Assembly Machine	**B1**	**B2**	**B3**
	2.3	3.7	3.1
A1	3.4	2.8	3.2
	3.5	3.7	3.5
	3.5	3.9	3.3
A2	2.6	3.9	3.4
	3.6	3.4	3.5
	2.4	3.5	2.6
A3	2.7	3.2	2.6
	2.8	3.5	2.5

Decide whether an appropriate model has random-effects, mixed-effects or fixed-effects and derive the appropriate ANOVA table.

State clearly any conclusions that may be drawn at the 5% level of significance.

Your solution

Do the calculations on separate paper and use the space here and on the following page for your summary and conclusions.

Your solution contd.

Answer

Both the machines and the testing stations are effectively chosen at random from populations consisting of all possible types so that their influence is described by a random-effects model. The calculations necessary to find MS_A, MS_B, MS_{AB} and MS_E are shown below.

	B1	**B2**	**B3**	**Totals**
A1	2.3 3.4 3.5	3.7 2.8 3.7	3.1 3.2 3.5	
	$T_{11} = 9.2$	$T_{12} = 10.2$	$T_{13} = 9.8$	$T_{1\cdot\cdot} = 29.2$
A2	3.5 2.6 3.6	3.9 3.9 3.4	3.3 3.4 3.5	
	$T_{21} = 9.7$	$T_{22} = 11.2$	$T_{23} = 10.2$	$T_{2\cdot\cdot} = 31.1$
A3	2.4 2.7 2.8	3.5 3.2 3.5	2.6 2.6 2.5	
	$T_{31} = 7.9$	$T_{32} = 10.2$	$T_{33} = 7.7$	$T_{3\cdot\cdot} = 25.8$
Totals	$T_{\cdot 1\cdot} = 26.8$	$T_{\cdot 2\cdot} = 31.6$	$T_{\cdot 3\cdot} = 27.7$	$T_{\cdots} = 86.1$

$a = 3,\ b = 3,\ n = 3,\ N = 27$ and the sums of squares calculations are:

$$SS_T = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk}^2 - \frac{T_{\cdots}^2}{N} = 2.3^2 + 3.4^2 + \ldots + 2.6^2 + 2.5^2 - \frac{86.1^2}{27} = 5.907$$

$$SS_A = \sum_{i=1}^{3} \frac{T_{i\cdot\cdot}^2}{bn} - \frac{T_{\cdots}^2}{N} = \frac{29.2^2 + 31.1^2 + 25.8^2}{3 \times 3} - \frac{86.1^2}{27} = 1.602$$

$$SS_B = \sum_{j=1}^{3} \frac{T_{\cdot j\cdot}^2}{an} - \frac{T_{\cdots}^2}{N} = \frac{26.8^2 + 31.6^2 + 27.7^2}{3 \times 3} - \frac{86.1^2}{27} = 1.447$$

$$SS_{AB} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{T_{ij\cdot}^2}{n} - \frac{T_{\cdots}^2}{N} - SS_A - SS_B$$

$$= \frac{9.2^2 + 10.2^2 + \ldots + 10.2^2 + 7.7^2}{3} - \frac{86.1^2}{27} - 1.602 - 1.447 = 0.398$$

$$SS_E = SS_T - SS_A - SS_B - SS_{AB} = 5.907 - 1.602 - 1.447 - 0.398 = 2.46$$

Answer continued

The results are combined in the following ANOVA table

Source	SS	DoF	MS	F (Random)	F (Random)
Factor $\boldsymbol{A}$ (Machines)	1.602	2	0.801	$\frac{MS_A}{MS_{AB}}$	8.05 $F_{2,4} = 6.94$
Factor $\boldsymbol{B}$ (Stations)	1.447	2	0.724	$\frac{MS_B}{MS_{AB}}$	7.28 $F_{2,4} = 6.94$
Interaction $\boldsymbol{AB}$	0.398	4	0.099(5)	$\frac{MS_{AB}}{MS_E}$	0.728 $F_{4,18} = 2.93$
Residual $\boldsymbol{E}$	2.460	18	0.136		
Totals	5.907	26			

The following conclusions may be drawn.

Interaction: There is insufficient evidence to support the hypothesis that interaction takes place between the factors.

Factor A: Since $8.05 > 6.94$ we have sufficient evidence to reject the hypothesis that the assembly machines do not affect the assembly characteristic.

Factor B: Since $7.28 > 6.94$ we have sufficient evidence to reject the hypothesis that the choice of testing station does not affect the assembly characteristic.

3. Two-way ANOVA versus one-way ANOVA

You should note that a two-way ANOVA design is rather more efficient than a one-way design. In the last example, we could fix the testing station and look at the electronic assemblies produced by a variety of machines. We would have to replicate such an experiment for *every* testing station. It would be very difficult (impossible!) to exactly duplicate the same conditions for all of the experiments. This implies that the consequent experimental error could be very large. Remember also that in a one-way design we cannot check for interaction between the factors involved in the experiment. The three main advantages of a two-way ANOVA may be stated as follows:

(a) It is possible to simultaneously test the effects of two factors. This saves both time and money.

(b) It is possible to determine the level of interaction present between the factors involved.

(c) The effect of one factor can be investigated over a variety of levels of another and so any conclusions reached may be applicable over a range of situations rather than a single situation.

Exercises

1. The temperatures, in Celsius, at three locations in the engine of a vehicle are measured after each of five test runs. The data are as follows. Making the usual assumptions for a two-way analysis of variance without replication, test the hypothesis that there is no systematic difference in temperatures between the three locations. Use the 5% level of significance.

Location	Run 1	Run 2	Run 3	Run 4	Run 5
A	72.8	77.3	82.9	69.4	74.6
B	71.5	72.4	80.7	67.0	74.0
C	70.8	74.0	79.1	69.0	75.4

2. Waste cooling water from a large engineering works is filtered before being released into the environment. Three separate discharge pipes are used, each with its own filter. Five samples of water are taken on each of four days from each of the three discharge pipes and the concentrations of a pollutant, in parts per million, are measured. The data are given below. Analyse the data to test for differences between the discharge pipes. Allow for effects due to pipes and days and for an interaction effect. Treat the pipe effects as fixed and the day effects as random. Use the 5% level of significance.

Day	Pipe A				
1	160	181	163	173	178
2	175	170	219	166	171
3	169	186	179	178	183
4	230	206	216	195	250
Day	**Pipe B**				
1	172	164	186	185	172
2	177	170	156	140	155
3	193	194	189	156	181
4	212	235	195	206	209
Day	**Pipe C**				
1	214	196	207	219	200
2	186	184	181	189	179
3	209	220	199	185	228
4	254	293	283	262	259

Answers

1. We calculate totals as follows.

Run	Total	Location	Total
1	215.1	A	377.0
2	223.7	B	365.6
3	242.7	C	368.3
4	205.4	Total	1110.9
5	224.0		
Total	1110.9		

$$\sum\sum y_{ij}^2 = 82552.17$$

The total sum of squares is

$$8255217 - \frac{1110.9^2}{15} = 278.916 \quad \text{on } 15 - 1 = 14 \text{ degrees of freedom.}$$

The between-runs sum of squares is

$$\frac{1}{3}(215.1^2 + 223.7^2 + 242.7^2 + 205.4^2 + 224.0^2) - \frac{1110.9^2}{15} = 252.796$$

on $5 - 1 = 4$ degrees of freedom.

The between-locations sum of squares is

$$\frac{1}{5}(377.0^2 + 365.6^2 + 368.3^2) - \frac{1110.9^2}{15} = 14.196 \quad \text{on } 3 - 1 = 2 \text{ degrees of freedom.}$$

By subtraction, the residual sum of squares is

$$278.916 - 252.796 - 14.196 = 11.924 \quad \text{on } 14 - 4 - 2 = 8 \text{ degrees of freedom.}$$

The analysis of variance table is as follows.

Source of variation	Sum of squares	Degrees of freedom	Mean square	Variance ratio
Runs	252.796	4	63.199	
Locations	14.196	2	7.098	4.762
Residual	11.924	8	1.491	
Total	278.916	14		

The upper 5% point of the $F_{2,8}$ distribution is 4.46. The observed variance ratio is greater than this so we conclude that the result is significant at the 5% level and reject the null hypothesis at this level. The evidence suggests that there are systematic differences between the temperatures at the three locations. Note that the Runs mean square is large compared to the Residual mean square showing that it was useful to allow for differences between runs.

Answers continued

2. We calculate totals as follows.

	Day 1	Day 2	Day 3	Day 4	Total
Pipe A	855	901	895	1097	3748
Pipe B	879	798	913	1057	3647
Pipe C	1036	919	1041	1351	4347
Total	2770	2618	2849	3505	11742

$$\sum\sum\sum y_{ijk}^2 = 2356870$$

The total number of observations is $N = 60$.

The total sum of squares is

$$2356870 - \frac{11742^2}{60} = 58960.6$$

on $60 - 1 = 59$ degrees of freedom.

The between-cells sum of squares is

$$\frac{1}{5}(855^2 + \cdots + 1351^2) - \frac{11742^2}{60} = 58960.6$$

on $12 - 1 = 11$ degrees of freedom, where by "cell" we mean the combination of a pipe and a day.

By subtraction, the residual sum of squares is

$$58960.6 - 48943.0 = 10017.6$$

on $59 - 11 = 48$ degrees of freedom.

The between-days sum of squares is

$$\frac{1}{15}(2770^2 + 2618^2 + 2849^2 + 3505^2) - \frac{11742^2}{60} = 30667.3$$

on $4 - 1 = 3$ degrees of freedom.

The between-pipes sum of squares is

$$\frac{1}{20}(3748^2 + 3647^2 + 4347^2) - \frac{11742^2}{60} = 14316.7$$

on $3 - 1 = 2$ degrees of freedom.

By subtraction the interaction sum of squares is

$$48943.0 - 30667.3 - 14316.7 = 3959.0$$

on $11 - 3 - 2 = 6$ degrees of freedom.

Answers continued

The analysis of variance table is as follows.

Source of variation	Sum of squares	Degrees of freedom	Mean square	Variance ratio
Pipes	14316.7	2	7158.4	10.85
Days	30667.3	3	10222.4	48.98
Interaction	3959.0	6	659.8	3.16
Cells	48943.0	11	4449.4	21.32
Residual	10017.6	48	208.7	
Total	58960.6	59		

Notice that, because Days are treated as a random effect, we divide the Pipes mean square by the Interaction mean square rather than by the Residual mean square.

The upper 5% point of the $F_{6,48}$ distribution is approximately 2.3. Thus the Interaction variance ratio is significant at the 5% level and we reject the null hypothesis of no interaction. We must therefore conclude that there are differences between the means for pipes and for days and that the difference between one pipe and another varies from day to day. Looking at the mean squares, however, we see that both the Pipes and Days mean squares are much bigger than the Interaction mean square. Therefore it seems that the interaction effect is relatively small compared to the differences between days and between pipes.