Contents 47

Mathematics and Physics Miscellany

Dimensional Analysis in Engineering

47.1

Introduction

Very often, the calculations carried out in engineering and science lead to a result which is not just a pure number but rather a number with a physical meaning, for example so many metres per second or so many kilograms per cubic metre or so many litres. If the numbers which are the input to the calculation also have a physical meaning then we can bring into play a form of analysis (called dimensional analysis) which supplements the arithmetical part of the calculation with an extra analysis of the physical units associated with the numerical values. In its simplest form dimensional analysis just helps in checking that a formula makes sense. For example, consider the *constant acceleration* equation

$$s = ut + \frac{1}{2}at^2$$

On the left-hand side s represents a distance in (say) *metres*. On the right-hand side a being acceleration is so many *metres per second per second*, which, when multiplied by t^2 which is *seconds squared* gives *metres*, as does the other term ut. Thus if somebody copied down the equation wrongly and wrote $\frac{1}{2}at$ (instead of $\frac{1}{2}at^2$) then we could surmise that there must be a mistake. This simple example illustrates the checking of an equation for dimensional homogeneity. However, it turns out that this passive **checking** aspect of dimensional analysis can be enhanced to give a useful **deductive** aspect to the method, so that it is possible to go some way towards finding out useful information about the equations which describe a system even when they are not fully known in advance. The following pages give many examples and exercises which will show how far it is possible to progress by adding an analysis of physical units to the usual arithmetic of numbers.

Prerequisites

Before starting this Section you should . . .

- be fully familiar with the laws of indices and with negative and fractional indices
- be able to solve sets of linear equations involving up to three variables

Learning Outcomes

On completion you should be able to . . .

- check the dimensional validity of a given equation
- determine which combinations of physical variables are likely to be important in the equations which describe a system's behaviour

1. Introduction to dimensional analysis

It is well known that it does not make sense to **add** together a mass of, say 5 kg and a time of 10 seconds. The same applies to **subtraction**. Mass, length and time are different from one another in a fundamental way but it is possible to **divide** length by time to get a **velocity**. It is also possible to **multiply** quantities, for example length multiplied by length gives **area**.

We note that, strictly speaking, we multiply a **number** of metres by a **number** of metres to get a **number** of square metres; however the terminology 'length times length equals area' is common in texts dealing with dimensional analysis.

We use the symbol [ ] to indicate the **dimensions** of a quantity:

$$[\text{mass}] = \mathsf{M} \qquad [\text{length}] = \mathsf{L} \qquad [\text{time}] = \mathsf{T}$$

The dimensions of other quantities can be written in terms of M, L and T using the definition of the particular quantity. For example:

$$[\text{area}] = [\text{length} \times \text{length}] = \mathsf{L}^2$$

$$[\text{density}] = \left[\frac{\text{mass}}{\text{volume}}\right] = \frac{\mathsf{M}}{\mathsf{L}^3} = \mathsf{ML}^{-3}$$

Some 'rules' can help in calculating dimensions:

Rule 1. Constant numbers in formulae do not have a dimension and are ignored.

Rule 2. Angles in formulae do not have a dimension and are ignored.

Newton's second law relates force, mass and acceleration, and we have:

$$[\text{force}] = [\text{mass} \times \text{acceleration}] = [\text{mass}] \times [\text{acceleration}] = \mathsf{MLT}^{-2}$$

$$[\text{work}] = [\text{force} \times \text{distance}] = \mathsf{MLT}^{-2} \times \mathsf{L} = \mathsf{ML}^2\mathsf{T}^{-2}$$

$$[\text{kinetic energy}] = \left[\frac{1}{2}\,\text{mass} \times (\text{velocity})^2\right] = \mathsf{M}(\mathsf{LT}^{-1})^2 = \mathsf{ML}^2\mathsf{T}^{-2}$$

Note that the sum and product rules for indices have been used here and that the dimensions of 'work' and 'kinetic energy' are the same.

Rule 3. In formulae only physical quantities with the same dimensions may be added or subtracted.

That is the end of the theory - the rules. Next is a list of the dimensions of useful quantities in mechanics. All come from the definition of the relevant quantity.

$[\text{mass}] = \mathsf{M}$
$[\text{length}] = \mathsf{L}$
$[\text{time}] = \mathsf{T}$
$[\text{area}] = [\text{length} \times \text{length}] = \mathsf{L}^2$
$[\text{volume}] = [\text{area} \times \text{length}] = \mathsf{L}^3$
$[\text{density}] = [\text{mass/volume}] = \mathsf{M}/\mathsf{L}^3 = \mathsf{ML}^{-3}$
$[\text{velocity}] = [\text{length/time}] = \mathsf{L}/\mathsf{T} = \mathsf{LT}^{-1}$
$[\text{acceleration}] = [\text{velocity/time}] = \mathsf{LT}^{-1}/\mathsf{T} = \mathsf{LT}^{-2}$
$[\text{force}] = [\text{mass} \times \text{acceleration}] = \mathsf{MLT}^{-2}$
$[\text{moment of force, torque}] = \mathsf{MLT}^{-2} \times \mathsf{L} = \mathsf{ML}^2\mathsf{T}^{-2}$
$[\text{impulse}] = [\text{force} \times \text{time}] = \mathsf{MLT}^{-2} \times \mathsf{T} = \mathsf{MLT}^{-1}$
$[\text{momentum}] = [\text{mass} \times \text{velocity}] = \mathsf{MLT}^{-1}$
$[\text{work}] = [\text{force} \times \text{distance}] = \mathsf{ML}^2\mathsf{T}^{-2}$
$[\text{kinetic energy}] = [\text{mass} \times (\text{velocity})^2] = \mathsf{ML}^2\mathsf{T}^{-2}$
$[\text{power}] = [\text{work/time}] = \mathsf{ML}^2\mathsf{T}^{-3}$

Task

State the dimensions, in terms of M, L and T of the quantities:

(a) pressure (force per unit area)
(b) line density (mass per unit length)
(c) surface density (mass per unit area)
(d) frequency (number per unit time)
(e) angular velocity (angle per unit time)
(f) angular acceleration (rate of change of angular velocity)
(g) rate of loss of mass (mass per unit time)

Your solution

Answer

(a) $\mathsf{ML}^{-1}\mathsf{T}^{-2}$ (b) ML^{-1} (c) ML^{-2} (d) T^{-1} (e) T^{-1} (f) T^{-2} (g) MT^{-1}

2. Simple applications of dimensional analysis

Application 1: Checking a formula

A piano wire has mass m, length l, and tension F. Which of the following formulae for the period of vibration, t, could be correct?

$$(a) \quad t = 2\pi\sqrt{\frac{ml}{F}} \qquad \text{or} \qquad (b) \quad t = 2\pi\sqrt{\frac{F}{ml}}$$

The dimensions of the expressions on the right-hand side of each formula are studied:

(a)
$$\begin{aligned}\left[2\pi\sqrt{\frac{ml}{F}}\right] &= [2\pi]\left[\frac{ml}{F}\right]^{\frac{1}{2}} \\ &= \left(\frac{\mathsf{ML}}{\mathsf{MLT}^{-2}}\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{\mathsf{T}^{-2}}\right)^{\frac{1}{2}} \\ &= (\mathsf{T}^2)^{\frac{1}{2}} \\ &= \mathsf{T}\end{aligned}$$

which **is consistent** with the dimensions of t, a period of vibration.

(b)
$$\begin{aligned}\left[2\pi\sqrt{\frac{F}{ml}}\right] &= [2\pi]\left[\frac{F}{ml}\right]^{\frac{1}{2}} \\ &= \left(\frac{\mathsf{MLT}^{-2}}{\mathsf{ML}}\right)^{\frac{1}{2}} \\ &= \left(\frac{\mathsf{T}^{-2}}{1}\right)^{\frac{1}{2}} \\ &= \mathsf{T}^{-1}\end{aligned}$$

which is **not consistent** with the dimensions of t, a period of vibration.

We can conclude that the second formula cannot be correct, whereas the first formula could be correct. (Actually, for case (b) we can see that the reciprocal of the studied quantity **would** be a time).

Task

Use the method of dimensions to check whether the following formulae are dimensionally valid where u and v are velocities, g is an acceleration, s is a distance and t is a time.

(a) $v^2 = u^2 + 2gs$ (b) $v = u - gt$

(c) $s = 0.5gt^2 + ut$ (d) $t = \sqrt{\dfrac{2s}{g}}$

Your solution

Answer

All are dimensionally valid.

Task

When a tension force T is applied to a spring the extension x is related to T by Hooke's law, $T = kx$, where k is called the stiffness of the spring. What is the dimension of k? Verify the dimensional correctness of the formula $t = 2\pi\sqrt{\dfrac{m}{k}}$ for the period of oscillation of a mass m suspended by a spring of stiffness k.

Your solution

Answer

Since T is a force, it has dimensions of [M][L][T]$^{-2}$.

So $k = T/x$ has dimensions [M][L][T^{-2}]/[L] = [M][T]$^{-2}$.

So $\sqrt{m/k}$ has dimensions of [M]$^{1/2}$/([M]$^{1/2}$[T]$^{-1}$) = [T]

Application 2: Deriving a formula

A simple pendulum has a mass m attached to a string of length l and is at a place where the acceleration due to gravity is g. We suppose that the periodic time of swing t of the pendulum depends jointly on $m,\ l$ and g and the aim is to find a formula for t in terms of $m,\ l$ and g which is dimensionally consistent.

Assume: $t =$ number $\times\ m^A l^B g^C$ sometimes written $t \propto m^A l^B g^C$

then: $[t] = [m^A l^B g^C]$ and $\mathsf{T} = \mathsf{M}^A \mathsf{L}^B (\mathsf{L}\mathsf{T}^{-2})^C = \mathsf{M}^A \mathsf{L}^B \mathsf{L}^C \mathsf{T}^{-2C} = \mathsf{M}^A \mathsf{L}^{B+C} \mathsf{T}^{-2C}$

so $\quad \mathsf{T} = \mathsf{M}^A \mathsf{L}^{B+C} \mathsf{T}^{-2C}$

For consistency we must have:

from M : $A = 0$

from T : $1 = -2C$

from L : $0 = B + C$

so $C = -\dfrac{1}{2}$ and $B = \dfrac{1}{2}$

Hence: $t =$ number $\times\ m^0 l^{\frac{1}{2}} g^{-\frac{1}{2}} =$ number $\times \sqrt{\dfrac{l}{g}}$ or $t \propto \sqrt{\dfrac{l}{g}}$

This is **not** a full proof of the pendulum formula and the method cannot be used to calculate the 2π constant in the pendulum formula. We shall give a more detailed discussion of this standard example in a later section.

Task

Use the method of dimensions to predict how:

(a) The tension T in a string may depend on the mass m of a particle which is being whirled round on its end in a circle of radius r and with speed v.

(b) The maximum height h reached by a stone may depend on its mass m, the energy E with which it is projected vertically and the acceleration due to gravity.

(c) The speed v of sound in a gas may depend on its pressure p, its density r and the acceleration g due to gravity.

Your solution

Answer

(a) $T \propto mv^2/r$ (b) $h \propto E/mg$ (c) $v \propto \sqrt{(p/r)}$

3. Finding the dimensional form of quantities

If we use the three basic dimensions mass, length and time, which we denote by the symbols M, L and T, then we shall need to find the dimensional formulae for quantities such as energy, angular momentum, etc., as we deal with a range of problems in mechanics. To do this we need to have a sufficient knowledge of the subject to enable us to recall important equations which relate the physical quantities which we are using in our mathematical theory. The dimensions of force and kinetic energy were derived briefly at the start of this section but to clarify the logical reasoning involved we now repeat the derivations in a slower step by step manner.

Perhaps the best-known equation in classical mechanics is the one expressing Newton's second law of particle motion (Force = mass $\times$ acceleration):

$$F = ma$$

To find the appropriate dimensional formula for force we can go through a sequence of operations as follows.

Step 1 Speed (or velocity) is usually quoted as so many metres per second or miles per hour and so its dimensional form is L/T or LT^{-1}.

Step 2 Acceleration is rate of change of velocity and so acceleration should be assigned the dimensional formula velocity/time, which leads to the result LT^{-2}.

Step 3 Using Newton's equation as given above we thus conclude that we obtain the expression for the dimensions of force by taking the product appropriate to mass $\times$ acceleration. Thus,

Force has the dimensions MLT^{-2}.

Example 1

Write down the formula for the kinetic energy of a particle of mass m moving with speed v and use it to decide what the dimensional expression for the energy should be.

Solution

The required formula is $E = \dfrac{1}{2}mv^2$ and so we obtain the dimensional expression for energy as:

Energy has the dimensions $\mathsf{M} \times (\mathsf{LT}^{-1})^2 = \mathsf{ML}^2\mathsf{T}^{-2}$.

4. Constants which have a dimensional expression

Several of the important equations which express physical laws include constants which have not only a numerical value (which can be looked up in a table of universal constants) but also have an appropriate dimensional formula. As a simple example, the "little g", g, used in the theory of falling bodies or projectiles might be given the value 32 feet per second per second or 981 cm per second per second (depending on the units used in our calculation). The shorthand terminology "32 feet per second squared" is often used in ordinary language and actually makes it clear that g is an acceleration, with the appropriate dimensional expression LT^{-2}.

Here are two examples of important equations which introduce constants which have both a numerical value and an associated dimensional expression. The first example is from quantum mechanics and the second is from classical mechanics.

1. In the quantum theory of light (relevant to the action of a modern laser) the energy of a light quantum for light of frequency ν is equal to $h\nu$, where h is called Planck's constant. This equation was formulated by Einstein in order to explain the behaviour of photoelectric emission, the physical process used in modern photoelectric cells. Planck's constant, h, which appears in this equation, later turned out to arise in much of quantum mechanics, including the theory of atomic structure.

2. The gravitational constant G of Newton's theory of universal gravitation appears in the equation which gives the gravitational force between two point masses M and m which are separated by a distance r:

$$F = \frac{GMm}{r^2}$$

Example 2

When quantum theory was applied to the theory of the hydrogen atom it was found that the motion of the electron around the nucleus was such that the angular momentum was restricted to be an integer multiple of the quantity $h/2\pi$, where h was the same Planck's constant which had arisen in the theory of radiation and of the photoelectric effect. Can you explain why this result is plausible by studying the dimensional formula for h?

Solution

We had earlier deduced that energy should be given the dimensional formula $\mathsf{ML}^2\mathsf{T}^{-2}$. Frequency is usually quoted as a number of cycles per second and so we should assign it the dimensional formula T^{-1} (on taking a pure number to be dimensionless and so to have a zero index for M, L and T). On taking h to be E/ν we conclude that h must be assigned the dimensional expression $\mathsf{ML}^2\mathsf{T}^{-2}/\mathsf{T}^{-1} = \mathsf{ML}^2\mathsf{T}^{-1}$. In particle mechanics the angular momentum is of the form "momentum $\times$ radius of orbit" i.e. mvr. This has the dimensions $\mathsf{M}(\mathsf{LT}^{-1})\mathsf{L} = \mathsf{ML}^2\mathsf{T}^{-1}$. Thus we see that Planck's constant, h, has the dimensions of an angular momentum. Dividing h by the purely numerical value 2π leaves this property unchanged. Indeed, if we rewrite the original equation $E = h\nu$ so as to use the angular frequency then we obtain $E = (h/2\pi)\omega$. The combination $h/2\pi$ is usually called "h bar" in quantum mechanics.

5. Further applications of dimensional analysis

Application 3: Deriving a formula

Almost every textbook dealing with dimensional analysis starts with the example of the simple pendulum and we did this in Application 2 a few pages back. We now look at this standard example in a slightly more general manner, using it to illustrate several useful methods and also to indicate the possible limitations in the use of dimensional analysis.

If we suppose that the periodic time t of a simple pendulum could possibly depend on the length l of the pendulum, the acceleration due to gravity g, and the mass m of the pendulum bob, then the usual approach is to postulate an equation of the form

$$t = \text{constant} \times m^A l^B g^C$$

Thus we concede from the start that the equation might include a multiplying factor (dimensionless) which is invisible to our approach via dimensional analysis. We now set out the related equation with the appropriate dimensional expression for each quantity

$$\mathsf{T} = \mathsf{M}^A \mathsf{L}^B (\mathsf{L}\mathsf{T}^{-2})^C = \mathsf{M}^A \mathsf{L}^{B+C} \mathsf{T}^{-2C}$$

We next look at M, L and T in turn and make their indices match up on the left and right of the equation. To do this we can think of the left-hand side as being $\mathsf{M}^0\mathsf{L}^0\mathsf{T}^1$. We find

(M) $\quad 0 = A$

(L) $\quad 0 = B + C$

(T) $\quad 1 = -2C$

giving $\quad A = 0,\ C = -1/2,\ B = 1/2.$

This leads to the conclusion that the periodic time takes the form $t = \text{constant} \times \sqrt{l/g}$.

This agrees with the result of simple mechanics (which also tells us that the numerical constant actually equals 2π). There are several points of interest here.

Comment 1

Although we were apparently looking at the problem of the simple pendulum our result simply shows that a time has the same dimensions as the square root of l/g. If we had started with the problem "If a particle starts from rest, how does the time t of fall depend on the distance fallen l, the gravitational acceleration g and the particle mass m?" we would have obtained the **same** result! The constant acceleration equations of simple mechanics give $l = \frac{1}{2}gt^2$ and so $t = \text{constant} \times \sqrt{l/g}$, where in this problem the constant has the value $\sqrt{2}$ instead of 2π.

Comment 2

The mass m does not enter into the final result, even though we included it in the list of variables which might possibly be relevant. With a more detailed knowledge of the problem we could have anticipated this by noticing that m does not appear in the differential equation which governs the motion of the pendulum:

$$l\frac{d^2\theta}{dt^2} = -g\sin(\theta)$$

On noting that the usual textbook theory replaces $\sin(\theta)$ by θ we arrive at Comment 3.

Comment 3
Our equation for t contains an unknown constant factor. This need **not** always be just a pure number, in the mathematical sense. For our pendulum problem we recall that the standard derivation of the equation for t involves the simplifying assumption that for a small angle θ we can set $\sin(\theta) = \theta$, where θ is the angle in radians. Thus we are admitting that the periodic time given by the usual textbook equation refers only to a **small** amplitude of swing. The basic equation which defines a radian is based on a ratio of distances; an angle of θ radians at the centre of a circle of radius r marks off an arc of length $r\theta$ along the circumference. Thus an angle in radians (or any other angular units) should be regarded as having dimensions $\mathsf{L}/\mathsf{L} = \mathsf{L}^0$ and is thus invisible in our process of dimensional analysis (just like a pure mathematical number). Thus for our simple pendulum problem we could set

$$t = F(\theta_0)\sqrt{l/g}$$

where F is some function of the amplitude θ_0 of the swing. Our dimensional analysis would not be able to tell us about F, which is invisible to the analysis. If we think about the physics of the problem then we can see that t should decrease with θ_0 and should be an even function (since θ_0 and $-\theta_0$ represent the same motion). We can thus reasonably suppose that the form of $F(\theta_0)$ starts off as $2\pi(1 - A\theta_0^2 \ldots)$, with A being a positive number. However, this detail is not visible to our dimensional analysis. This example highlights the fact that dimensional analysis is only an auxiliary tool and must always be backed up by an understanding of the system being studied. This need for a preliminary understanding of the system is evident right at the start of the analysis, since it is needed in order to decide which variables should be included in the list which appears in the postulated equation. An approach to the simple pendulum problem using a conservation of energy approach can be used; it leads to an expression for the periodic time which does indeed show that the periodic time is an even function of the amplitude θ_0 but it involves the evaluation of elliptic integrals, which are outside the range of this workbook.

Comment 4
Another way of writing the first simple result for the periodic time of the pendulum would be to use a dimensionless variable:

$$gt^2/l = \text{constant}$$

We now know that strictly speaking the right-hand side of this equation need be a constant only to the extent that it is dimensionless (so that it could depend on a dimensionless quantity such as the amplitude of the swing). The approach of using dimensionless variables to represent the equations governing the behaviour of a system is often useful in modelling experiments and is particularly associated with the work of Buckingham. The Buckingham Pi theorem is quoted in various forms in the literature. Its essential content is that if we start with n variables then we can always use our basic set of dimensional units (M, L and T in the pendulum case) to construct a set of dimensionless variables (fewer than n) which will describe the behaviour of the system. This idea is used in some of our later examples.

Planetary motion: Kepler's laws

Newton showed that the combination of his law of universal gravitation with his second law of motion would lead to the prediction that the planets should move in elliptic orbits around the Sun. Further, since the gravitational force is central (i.e. depends only on r and not on any angles) it follows that a planet moves in a plane and has constant angular momentum; this second fact is equivalent to the Kepler law which says that a planet "sweeps out equal areas in equal times". Another Kepler law relates the time t for a planet to go around the Sun to the mean distance r of the planet from the Sun (or rather to the major semi-axis of its elliptic orbit). To attack this problem using dimensional analysis we need to find the dimensional formula to assign to the gravitational constant G. We start from the equation given on page 9 for the inverse square law force and rewrite it to give

$$G = \frac{Fr^2}{mM}$$

So that we arrive at the dimensional formula associated with G:

G **has the dimensions** $(\mathsf{MLT}^{-2})\mathsf{L}^2\mathsf{M}^{-2} = \mathsf{ML}^3\mathsf{T}^{-2}$.

Example 3

Find how the periodic time t for a planetary orbit is related to the radius r of the orbit.

Solution

We suppose that t depends on G, on the Sun's mass M and on r. (Looking at the planet's equation of motion suggests that the planet's mass m can be omitted). We set

$$t = \text{constant} \times G^A M^B r^C$$

leading to the dimensional equation

$$\mathsf{T} = (\mathsf{M}^{-1}\mathsf{L}^2\mathsf{T}^{-2})^A \mathsf{M}^B \mathsf{L}^C$$

Comparing coefficients on both sides gives three equations:

(M) $\quad 0 = B - A$

(L) $\quad 0 = 3A + C$

(T) $\quad 1 = -2A$

which produces the result $A = -1/2,\ B = -1/2,\ C = 3/2$.

If we use t^2 instead of t we conclude that t^2 is proportional to $\dfrac{r^3}{GM}$. This is in accord with the Kepler law which states that t^2/r^3 has the same value for all the planets.

The search title "Kepler's laws" in Google gives sites with much detail about the motion both of planets around the Sun and of satellites around the planets. The ratio t^2/r^3 varies by about one percent across all the planets. (See the following Comment)

Comment

You might have noticed that we used two laws due to Newton in the calculation above but ignored his third law which deals with action and reaction. The Sun is acted on by an equal and opposite force to that which it exerts on the planet. Thus the Sun should move as well! In principle it goes round a very small orbit. A detailed mathematical analysis shows that for the special case of one sun of mass M and one planet of mass m then the correct mass to use in the calculation above is the reduced mass given by the formula $mM/(m+M)$. This has the dimensions of a mass and thus our simple dimensional analysis cannot distinguish it from M. The modified Kepler's law then predicts that r^3/t^2 should be proportional to $(M+m)$ and the planets obey this modified law almost perfectly.

This again illustrates the need for as much detailed knowledge as possible of a system to back up the use of dimensional analysis. In the real solar system, of course, there are several planets and they exert weak gravitational forces on one another. Indeed the existence of the planet Neptune was surmised because of the small deviations which it produced in the motion of the planet Uranus.

An example from electrostatics and atomic theory

In the electrostatic system of units, the force F acting between two charges q and Q which are a distance r apart is given by an inverse square law of the form

$$F = \text{constant} \times \frac{qQ}{r^2}, \qquad \text{where the constant} = 1 \text{ or } 4\pi \text{ depending on the units used.}$$

If we wish to decide on an appropriate dimensional formula to represent charge then we set $Q = \mathsf{M}^A\mathsf{L}^B\mathsf{T}^C$ in the dimensional equation which represents this inverse square law equation. We have

$$\mathsf{MLT}^{-2} = (\mathsf{M}^A\mathsf{L}^B\mathsf{T}^C)^2\mathsf{L}^{-2}$$

Comparing coefficients on both sides gives the equations

(M) $\quad 1 = 2A$

(L) $\quad 1 = 2B - 2$

(T) $\quad -2 = 2C$

Thus $A = 1/2,\ B = 3/2,\ C = -1$, so Q^2 has the dimensional formula $\mathsf{ML}^3\mathsf{T}^{-2}$. ($Q^2$ is simpler to use than Q and appears in most other results.)

Example 4

In the theory of the hydrogen atom the energy levels of the atom depend on the principal quantum number $n = 1, 2, 3, \ldots$ and equal $\dfrac{-1}{2n^2}$ times an energy unit which depends on the electronic charge e, the electron mass m and Planck's constant h (which we have seen in an earlier problem to have the same dimensions as an angular momentum). Use dimensional analysis to find how the atomic energy unit (the Rydberg) depends on $e,\ m$ and h.

Solution

We require that the combination $e^{2A}m^Bh^C$ should have the dimensions of an energy.

In terms of dimensional formulae we thus obtain the equation

$$\mathsf{ML}^2\mathsf{T}^{-2} = (\mathsf{ML}^3\mathsf{T}^{-2})^A\mathsf{M}^B(\mathsf{ML}^2\mathsf{T}^{-1})^C$$

Comparing coefficients on both sides gives the equations

(M) $\quad 1 = A + B + C$

(L) $\quad 2 = 3A + 2C$

(T) $\quad -2 = -2A - C$

Adding twice the (T) equation to the (L) equations shows that $A = 2$ and then the other variables are found to be $B = 1$ and $C = -2$. Thus we find that the Rydberg atomic energy unit must depend on $e,\ m$ and h via the combination me^4/h^2, which is as given by the full theory. Indeed, the quantities $e,\ m$ and h are the quantities appearing in the Schrödinger equation for the hydrogen atom and are thus the appropriate ones to use.

Some examples from the theory of fluids

In fluid theory the concepts of viscosity and of surface tension play an important role and so we must decide on the appropriate dimensional formula to assign to the quantities which are associated with these concepts. The coefficient of viscosity μ of a fluid is such that the force per unit area across a fluid plane is μ times the fluid velocity gradient perpendicular to that surface. (We note that it is an assumption based on a combination of theory and experience to assert that such a well defined coefficient does exist; in general a fluid is called a Newtonian fluid if it has a well defined μ. Even so, μ can vary with temperature and other variables).

If we set force/area $= \mu \times$ velocity/distance then we find

$$\mu = (\text{force} \times \text{distance})/(\text{area} \times \text{velocity}) = (\mathsf{MLT}^{-2} \times \mathsf{L})/(\mathsf{L}^2 \times \mathsf{LT}^{-1}) = \mathsf{ML}^{-1}\mathsf{T}^{-1}$$

Example 5

Suppose that a sphere of radius r moves at a slow speed v through a fluid with coefficient of viscosity μ and density ρ. Find the way in which the resistive force acting on the sphere should depend on these three quantities in two different ways:

(a) by omitting ρ from the list of variables

(b) by omitting μ from the list of variables

Solution

(a) We set $F = \text{constant} \times \mu^A r^B v^C$ and so arrive at the dimensional equation

$$\mathsf{MLT}^{-2} = (\mathsf{ML}^{-1}\mathsf{T}^{-1})^A \mathsf{L}^B (\mathsf{LT}^{-1})^C$$

Comparing coefficients on both sides gives

(M) $\quad 1 = A$

(L) $\quad 1 = B + C - A$

(T) $\quad -2 = -A - C$

which yields the result $A = 1,\ B = 1,\ C = 1.$

Thus the resistance is proportional to the product $rv\mu$. Detailed theory gives this result (called Stokes' law) with the constant equal to 6π. This formula is appropriate for very slow motion.

We now include the density ρ in the analysis and omit μ.

(b) On supposing that the resistive force is given by $F = r^A v^B \rho^C$ we obtain the dimensional equation

$$\mathsf{MLT}^{-1} = (\mathsf{L})^A (\mathsf{LT}^{-1})^B (\mathsf{ML}^{-3})^C$$

leading to the three equations:

(M) $\quad 1 = C$

(L) $\quad 1 = A + B - 3C$

(T) $\quad -2 = -B$

These equations give the results $A = 2, B = 2, C = 1$, so that the force is calculated to vary as $\rho r^2 v^2$. This expression for the resistance is appropriate for speeds which are not too high but at which the moving object has to push the fluid aside (thus giving it some energy related to ρv^2) while the Stokes' force of Solution (a) refers to motion so slow that this dynamical effect is negligible. In the theory of flight the force on an aircraft's wing has two components, lift and drag. These depend on the aircraft's speed and on the angle of inclination of the wing. The expression $\rho r^2 v^2$ can be interpreted as being area$\times\rho v^2$ for a wing with a given surface area and the numerical multiplying factor which gives the actual drag and lift as a function of the wing angle of inclination is expressed as a lift or drag coefficient.

Comment

In Solutions (a) and (b) we have obtained two different expressions with the dimensions of a force, by selecting three variables at a time from a list of four variables. We can clearly obtain a dimensionless quantity by taking the ratio of the two different force expressions. This gives us

$$\rho r^2 v^2/(r\mu v) = \rho r v/\mu$$

This dimensionless quantity is called the Reynold's number and it appears in various parts of the theory of fluids. In this example r was the radius of a moving spherical object; in other cases it can be the radius of a pipe through which the fluid is flowing. Experiments show that (in both cases) the fluid flow is smooth at low Reynold's numbers but becomes turbulent above a critical Reynold's number (the numerical value of which depends on the particular system being studied). If we adopt the point of view which writes the results of dimensional analysis in the form of equations which involve dimensionless variables then the results of problems (a) and (b) can be written in a form which involves the Reynold's number:

$$F/(\rho v^2 r^2) = f(\rho r v/\mu)$$

Where f is some function of the Reynold's number. If $f(x)$ starts off as $6\pi/x$ and tends to a constant for large v then we will obtain the two special cases (a) and (b) which we treated in the problems. If you are familiar with the history of flight you might know that if v is sufficiently large for the compressibility of the fluid to play an important role then f should be extended to involve another dimensionless variable, the Mach number, equal to v/v_s where v_s is the speed of sound.

Example 6

Suppose that a fluid has the viscosity coefficient μ and flows through a tube with a circular cross section of radius r and length l. If the pressure difference across the ends of the tube is p, find a possible expression for the volume per second of fluid passing through a tube, using pressure gradient p/l as a variable.

Solution

We set volume per second = constant $\times\ r^A \times \left(\dfrac{p}{l}\right)^B \times \mu^C$ and obtain the dimensional equation

$$\mathsf{L}^3\mathsf{T}^{-1} = \mathsf{L}^A(\mathsf{ML}^{-2}\mathsf{T}^{-2})^B(\mathsf{ML}^{-1}\mathsf{T}^{-1})^C$$

leading to the equations

(M) $\quad 0 = B + C$

(L) $\quad 3 = A - 2B - C$

(T) $\quad -1 = -2B - C$

The result is found to be $A = 4, B = 1, C = -1$, thus leading to the conclusion that the volume per second should vary as $r^4\mu^{-1}pl^{-1}$. This is the result given by a theoretical treatment which allows for the fact that the fluid sticks to the walls and so has a speed which increases from zero to a maximum as one moves from the wall to the centre of the tube.

A problem involving surface tension

If we recall that the surface tension coefficient σ can be regarded as an energy per unit area of a liquid surface then we can conclude that σ has the dimensions $\mathsf{ML}^2\mathsf{T}^{-2}/\mathsf{L}^2 = \mathsf{MT}^{-2}$.

Example 7

Consider a wave of wavelength λ travelling on the surface of a fluid which has the density ρ and the surface tension coefficient σ. Including the acceleration due to gravity g in the list of relevant variables, derive possible formulae for the velocity of the wave as a function of λ by

(a) omitting σ from the list,   (b) omitting g from the list.

Solution

(a) The equation $v =$constant $\times\ \lambda^A\rho^B g^C$ leads to the dimensional equation

$$\mathsf{LT}^{-1} = \mathsf{L}^A(\mathsf{ML}^{-3})^B(\mathsf{LT}^{-2})^C$$

which gives the linear equations

(M) $\quad 0 = B$

(L) $\quad 1 = A - 3B + C$

(T) $\quad -1 = -2C$

The solution is quickly seen to be $A = \frac{1}{2}, B = 0, C = \frac{1}{2}$. We find that v^2 is proportional to $g\lambda$.

(b) If we set $v =$constant $\times\ \lambda^A\sigma^B\rho^C$ then we obtain the dimensional equation

$$\mathsf{LT}^{-1} = \mathsf{L}^A(\mathsf{MT}^{-2})^B(\mathsf{ML}^{-3})^C$$

leading to the linear equations

(M) $\quad 0 = B + C$

(L) $\quad 1 = A - 3C$

(T) $\quad -1 = -2B$

This gives the solution $A = -\frac{1}{2}, B = \frac{1}{2}, C = -\frac{1}{2}$. We find that v^2 should be proportional to $\sigma/(\lambda\rho)$.

We note that the results (a) and (b) depend on the wavelength λ in different ways, which might suggest that (a) is more relevant for long waves and (b) is more relevant for short waves. Detailed solution of the fluid equations of motion gives the result

$$v^2 = g\lambda/(2\pi) + 2\pi\sigma/(\lambda\rho)$$

showing that long waves involve a kind of up and down oscillation of the water mass under the action of gravity while short waves involve a strong curvature of the surface which brings surface tension forces into play.

Comment

The preceding examples dealing with fluids both involve situations which are more complicated than the simple pendulum problem which was treated earlier. In particular, both the fluid problems are such that if we use the three basic dimensions M, L and T then we can form **two** different dimensionless quantities from the list of physical variables which are regarded as important for the problem being studied. Thus, if we take the traditional simple approach we can obtain two different possible results by removing one variable at a time. If we use the Buckingham approach based on dimensionless variables then we arrive at an equation which says that the first dimensionless quantity is some arbitrary function of the second one.

An example involving temperature

The van der Waals' equation of state was an early equation which tried to describe the way in which the behaviour of a real gas differs from that of an ideal gas. One mole of an ideal gas obeys the equation $PV = RT$, where P, V and T are the pressure and volume of the gas and T is the absolute temperature. R is the universal gas constant. The simple equation of state can be derived from kinetic theory by assuming that the gas atoms are of negligible size and that they move independently. van der Waals' equation introduced two parameters a and b into the equation of state; his modification of the ideal gas equation takes the form

$$(P + aV^{-2})(V - b) = RT$$

The parameter a is intended to allow roughly for the effects of interactions between the atoms while b allows for the finite volume of the atoms. When we attempt to apply the methods of dimensional analysis to this equation we run into a problem of notation. We have already used the symbol T to represent time and so must not confuse that symbol with the temperature T in the traditional theory of gases. Since the absolute temperature is often quoted as "degrees Kelvin" we shall adopt the symbol K for temperature. The product PV in the ideal gas equation has the dimensions of an energy, since (force per unit area) x (volume) = force x distance = energy. If we now use a symbol K for a new dimensional quantity, the temperature, then we conclude that the gas constant R is a dimensional constant, with the dimensional formula given by

$$[\text{Energy}/K] = \mathsf{ML}^2\mathsf{T}^{-2}\mathsf{K}^{-1}$$

Inspection of the van der Waals' equation makes it clear that if we re-arrange it so as to find the volume V of the gas at a given pressure and temperature then we arrive at a **cubic** equation for V. For some P and T values this equation has one real and two complex conjugate roots and so it is reasonable to take the real root as being the one which represents the actual physical volume of the gas. However, there are regions in which the cubic equation has three real roots, which sets us a problem of interpretation! Various workers have tackled this problem by trying to apply the principles of thermodynamics (in particular, in attempting to find the solution with the minimum free energy) and their conclusion is that the "triple real solution" region is actually one in which the gas is condensing into a liquid (so that the volume contains both gas and liquid portions). If the temperature is sufficiently high, above what is called the critical temperature, then the gas cannot be made to condense even when the pressure is increased.

Example 8

Work out the dimensional formulae for the parameters a and b in the van der Waals' equation and then find a possible equation which gives the critical temperature of the gas in terms of a, b and R.

Solution

In van der Waals' equation $(P + aV^{-2})(V - b) = RT$ the term aV^{-2} is in the bracket with the pressure P and so must itself have the dimensions of a pressure i.e. a force per unit area. Thus we conclude that a has the dimensions of
[force] x [squared volume]/[area], which gives the result

$$(\mathsf{MLT}^{-2}) \times \mathsf{L}^6/\mathsf{L}^2 = \mathsf{ML}^5\mathsf{T}^{-2}$$

b clearly has the dimensions L^3 of a volume. We have already obtained the dimensions of R in our previous discussion and so we try the following expression for the critical temperature:

$$K = a^A b^B R^C$$

The dimensional equation corresponding to our proposed equation is

$$\mathsf{K} = (\mathsf{ML}^5\mathsf{T}^{-2})^A(\mathsf{L}^3)^B(\mathsf{ML}^2\mathsf{T}^{-2}K^{-1})^C$$

This gives four linear equations

(M) $\quad 0 = A + C$

(L) $\quad 0 = 5A + 3B + 2C$

(T) $\quad 0 = -2A - 2C$ (this is the same as the M equation)

(K) $\quad 1 = -C$

There thus turn out to be only three independent equations and these have the solution

$$A = 1, B = -1, C = -1.$$

Thus we conclude that the critical temperature should be a constant times $a/(bR)$, as is concluded by the detailed theory, which gives the value $8/27$ for the constant. The critical temperature turns out to be the temperature associated with the special P-V trajectory along which the cubic equation for V has a triple real root at one point, which is called the triple point.

6. Water flowing in a pipe

Introduction

One way of measuring the flow rate of water in a pipe is to measure the pressure drop across a circular plate, containing an orifice (hole) of known dimensions at its centre, when the plate is placed across the pipe. The resulting device is known as an orifice plate flow meter. The theory of fluid mechanics predicts that the volumetric flow rate, Q (m^3 s^{-1}), is given by

$$Q = C_d A_o \sqrt{\frac{2g\Delta h}{1 - A_o^2/A_p^2}}$$

where Δh (m) is the pressure drop (i.e. the difference in 'head' of water across the orifice plate) A_o (m^2) and A_p (m^2) are the areas of the orifice and the pipe respectively, g is the acceleration due to gravity (m s^{-2}) and C_d is a discharge coefficient that depends on viscosity and the flow conditions.

Problem in words

(a) Show that C_d is dimensionless.

(b) Rearrange the equation to solve for the area of the orifice, A_o, in terms of the other variables.

(c) Calculate the orifice diameter required if the head difference across the orifice plate is to be 200 mm when a volumetric flow rate of 100 cm^3 s^{-1} passes through a pipe with 10 cm inside diameter. Assume a discharge coefficient of 0.6 and take $g = 9.81$ m s^{-2}.

Mathematical analysis

(a) The procedure for checking dimensions is to replace every quantity by its dimensions (expressed in terms of length L, time T and mass M) and then equate the powers of the dimensions on either side of the equation. Q, the volume flow rate has dimensions m^3 s^{-1} which can be written L^{3}T^{-1}. Areas A_o and A_p have dimensions of m^2 which can be written L^2. g has dimensions m s^{-2} which can be written L T^{-2}. None of these involve the mass dimension. Consider the square root on the right-hand side of the equation. First look at the denominator. Since areas A_o and A_p have the same dimensions, A_o^2/A_p^2 is dimensionless. The number 1 is dimensionless, so the denominator is dimensionless. Now consider the numerator. The product $g\Delta h$ has dimensions LT^{-2}L = L^{2}T^{-2} . This means that the square root has dimensions LT^{-1}. After multiplying by an area (A_o) the dimensions are L^{3}T^{-1}. These are the dimensions of Q. So the factors other than C_d on the right-hand side of the equation together have the same dimensions as Q. This means that C_d must be dimensionless.

(b) Starting from the given equation $\quad Q = C_d A_o \sqrt{\dfrac{2g\Delta h}{1 - A_o^2/A_p^2}}$

Square both sides: $\quad Q^2 = C_d^2 A_o^2 \dfrac{2g\Delta h}{1 - A_o^2/A_p^2}$

Now multiply through by $1 - A_o^2/A_p^2$. $\quad Q^2(1 - A_o^2/A_p^2) = 2g\Delta h C_d^2 A_o^2$

Collect together terms in A_o^2: $\quad Q^2 = A_o^2(2g\Delta h C_d^2 + Q^2/A_p^2)$

Divide through by the bracketed term on the right-hand side of this equation and square root both sides:

$$A_o = \frac{Q}{\sqrt{2g\Delta h C_d^2 + Q^2/A_p^2}}$$

(c) Substitute $\Delta h = 0.2,\ A_p = 0.1,\ g = 9.81,\ Q = 10^{-3}$ and $C_d = 0.6$ in the above to give $A_o = 1.189 \times 10^{-3}$. The required orifice area is about 1.2×10^{-3} m^2.

7. Temperature as a dimensional variable

In Example 8 we introduced temperature as a new independent dimensional variable. The universal gas constant R will then acquire the dimensions of $\dfrac{[PV]}{[K]} = \dfrac{[\text{Energy}]}{[K]} = \text{M L}^2\ \text{T}^{-2}\ \text{K}^{-1}$, and other quantities in thermodynamic theory will acquire corresponding dimensional formulae.

Temperature is one of the fundamental entities on the International Units System, along with mass, length and time, which have played the major role in the examples which we have discussed. The standard symbol θ is often used for temperature, although we used K, since θ had been used to denote an angle in several examples. Example 4 involved a combination of electrostatics and dynamics and for the particular calculation treated it was sufficient to describe the electrical charge Q in terms of M, L and T. However, the International Units System has electrical current I as a basic dimensional quantity, so that charge Q becomes current times time, i.e. with dimensions IT. The potential difference between two points, being energy (i.e. work done) per unit charge would have dimensions given by $\text{ML}^2\ \text{T}^{-2}\ (\text{IT})^{-1} = \text{ML}^2\ \text{I}^{-1}\ \text{T}^{-3}$, whereas in the approach used in our Example 4 it would have the dimensions $\text{M}^{\frac{1}{2}}\ \text{L}^{\frac{1}{2}}\ \text{T}^{-1}$, since current would have had the dimensions $\text{M}^{\frac{1}{2}}\ \text{L}^{\frac{3}{2}}\ \text{T}^{-2}$. It is clear that in the (MLTI) system the resulting expressions are simpler than they would be if we attempted to do every calculation by reducing everything to the (MLT) system.

One basic difference between the approach of Example 4 and that which is adopted in Electrical Engineering (with I as a fundamental dimension) arises from the equation for the law of force between two charges. We took the force to be equal to a constant times the product of the two charges divided by the distance squared, whereas in modern approaches to electromagnetic theory our "constant" is taken to be a dimensional quantity (the reciprocal of the permittivity of free space).

Historically, the need for a set of dimensional quantities and a set of units which were jointly appropriate for both electrical and magnetic phenomena became more obvious with the publication of Maxwell's equations and the discovery of electromagnetic waves, in which the electrical and magnetic fields propagate together. This topic is too lengthy to be treated here but is discussed in detail in some of the works cited in subsection 8.

8. Some useful references

Nowadays there are many articles, books and web pages which deal with dimensional analysis and its applications.

8.1 Three Classic Books

(1) The book *Dimensional Analysis* by P.W. Bridgman is regarded as probably the most careful study of the power (and of the limitations) of dimensional analysis. It gives many applications but also stresses the importance of starting from a detailed knowledge of the governing equation of motion (often a differential equation) for a system in order to be able to make an intelligent choice of the variables which should appear in any expression which is to be analysed using dimensional analysis.

(2) The book *Hydrodynamics* by G. Birkhoff adopts a similar critical approach but deals specifically with problems involving fluid flow (where dimensionless quantities such as the Reynold's number play a role). Birkhoff uses some concepts from group theory in his discussion and gives the name "inspectional analysis" to the total investigative approach which combines dimensional analysis with other mathematical techniques and with physical insight in order to study a physical system.

(3) The book *Method of Dimensions* by A.W. Porter contains many examples from mechanics, thermodynamics and fluid theory, with discussion of both the power and the limitations of dimensional analysis.

Book (1) was published by Yale University press, (2) by Oxford University Press (in Britain) and (3) by Methuen. There should be a copy in most university libraries.

8.2 Some journal articles

The American journal of Physics has published several detailed explanations of the use of dimensional analysis together with particular examples. Here are a few:

- Volume 51, (1983) pages 137 to 140 by W.J. Remillard. *Applying Dimensional Analysis.*
- Volume 53, (1985) pages 549 to 552 by J.M. Supplee. *Systems of equations versus extended reference sets in dimensional analysis.* (This mentions the possibility of using separate L_x and L_y dimensions to replace the single L dimension for some special cases when the problem involves motion in two dimensions.)
- Volume 72, (2004) pages 534 to 537 by C.F. Bohren. *Dimensional analysis, falling bodies and the fine art of not solving differential equations.* (This gives a lengthy study of the theory of a body falling in the Earth's gravitational field, gradually introducing the effects of the Earth's finite radius, the Earth's rotation and the presence of air resistance.)
- Volume 71, (2003) pages 437 to 447 by J.F. Pryce. *Dimensional analysis of models and data sets.* (This article deals with the simple pendulum, with air resistance being discussed and with the example being used to illustrate general ideas about modelling and about the use of dimensionless variables.)

8.3 Web Sites

The site www.pigroup.de connects with a research group at the University of Stuttgart which uses dimensional analysis in various engineering applications. The site gives a list of the published work of the group.

8.4 Using Google

The simplest way to search nowadays is to type in an appropriate title in the Google search engine. To explore particular topics treated or mentioned in this section it is best to use titles such as:

Buckingham Pi Theorem (to see the different interpretations of this theorem).

Reynold's Number (to see the origin and uses of this number).

Wind Tunnel (to see details of modelling projects using dimensional analysis in the study of air resistance for various objects, including tennis balls).

Drag Coefficient (to see how the air resistance to a wing varies as the speed increases with particular reference to the lift and drag forces on a wing).

Van der Waals' Equation + **Critical Point** (this gives several pages which explain the theory of the van der Waals' equation and discuss how well it describes the behaviour of real gases; several pages also give alternative equations proposed by other workers).

Wikipedia Potential Difference

The topic touched on in Section 7, the use of the current I as a fundamental quantity in the dimensional analysis of problems involving electricity and magnetism, is an important one and so appears on several websites. The Google input Wikipedia Potential Difference leads to a useful table which sets out the dimensional expressions for many common quantities in terms of the (MLTI) system. The history of the development of the system of fundamental dimensions in electromagnetism is explained in the article by G. Thomas (Physics Education, Volume 14, page 116, 1979); this article is also obtained via Google.

You can, of course, try any other search title, if you have the patience to keep going until you find a particular site which gives you something useful!