Physics Case Studies

47.3

Introduction

This Section contains a compendium of case studies involving physics (or related topics) as an additional teaching and learning resource to those included in the previous Workbooks. Each case study may involve several mathematical topics; these are clearly stated at the beginning of each case study.

Prerequisites

Before starting this Section you should . . .

- have studied the Sections referred to at the beginning of each Case Study

Learning Outcomes

On completion you should be able to . . .

- appreciate the application of various mathematical topics to physics and related subjects

Physics Case Studies

Physics Case Study 1

Black body radiation 1

Mathematical Skills

Topic	Workbook
Logarithms and exponentials	[6]
Numerical integration	[31]

Introduction

A common need in engineering thermodynamics is to determine the radiation emitted by a body heated to a particular temperature at all wavelengths or a particular wavelength such as the wavelength of yellow light, blue light or red light. This would be important in designing a lamp for example. The total power per unit area radiated at temperature T (in K) may be denoted by $E(\lambda)$ where λ is the wavelength of the emitted radiation. It is assumed that a perfect absorber and radiator, called a **black body**, will absorb all radiation falling on it and which emits radiation at various wavelengths λ according to the formula

$$E(\lambda) = \frac{C_1}{\lambda^5\left[e^{C_2/(\lambda T)} - 1\right]} \qquad (1)$$

where $E(\lambda)$ measures the energy (in W m^{-2}) emitted at wavelength λ (in m) at temperature T (in K). The values of the constants C_1 and C_2 are 3.742×10^{-16} W m^{-2} and 1.439×10^{-2} m K respectively. This formula is known as Planck's distribution law. Figure 1.1 shows the radiation $E(\lambda)$ as a function of wavelength λ for various values of the temperature T. Note that both scales are plotted logarithmically. In practice, a body at a particular temperature is not a black body and its emissions will be less intense at a particular wavelength than a black body; the power per unit area radiated by a black body gives the ideal upper limit for the amount of energy emitted at a particular wavelength.

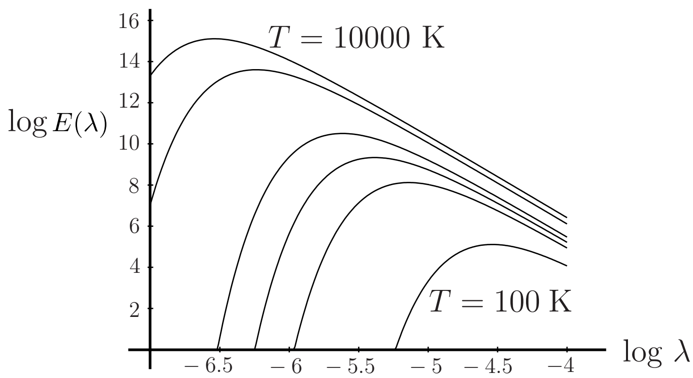

Figure 1.1

The emissive power per unit area $E(\lambda)$ plotted against wavelength (logarithmically) for a black body at temperatures of $T = 100$ K, 400 K, 700 K, 1500 K, 5000 K and 10000 K.

Problem in words

Find the power per unit area emitted for a particular value of the wavelength ($\lambda = 6 \times 10^{-7}$m). Find the temperature of the black body which emits power per unit area ($E(\lambda) = 10^{10}$ W m^{-2}) at a specific wavelength ($\lambda = 4 \times 10^{-7}$m)

Mathematical statement of problem

(a) A black body is at a temperature of 2000 K. Given formula (1), determine the value of $E(\lambda)$ when $\lambda = 6 \times 10^{-7}$m.

(b) What would be the value of T that corresponds to $E(\lambda) = 10^{10}$ W m^{-2} at a wavelength of $\lambda = 4 \times 10^{-7}$m (the wavelength of blue light)?

Mathematical analysis

(a) Here, $\lambda = 6 \times 10^{-7}$ and $T = 2000$. Putting these values in the formula gives

$$E(\lambda) = 3.742 \times 10^{-16} / \left(6 \times 10^{-7}\right)^5 / \left(e^{1.439 \times 10^{-2}/6 \times 10^{-7}/2000} - 1\right)$$

$$= 2.98 \times 10^{10} \text{ W m}^{-2} \text{ (to three significant figures).}$$

(b) Equation (1) can be rearranged to give the temperature T as a function of the wavelength λ and the emission $E(\lambda)$.

$$E(\lambda) = \frac{C_1}{\lambda^5\left[e^{C_2/(\lambda T)} - 1\right]} \qquad \text{so} \quad e^{C_2/(\lambda T)} - 1 = \frac{C_1}{\lambda^5 E(\lambda)}$$

and adding 1 to both sides gives

$$e^{C_2/(\lambda T)} = \frac{C_1}{\lambda^5 E(\lambda)} + 1.$$

On taking (natural) logs

$$\frac{C_2}{\lambda T} = \ln\left[\frac{C_1}{\lambda^5 E(\lambda)} + 1\right]$$

which can be re-arranged to give

$$T = \frac{C_2}{\lambda \ln\left[\dfrac{C_1}{\lambda^5 E(\lambda)} + 1\right]} \qquad (2)$$

Equation (2) gives a means of finding the temperature to which a black body must be heated to emit the energy $E(\lambda)$ at wavelength λ.

Here, $E(\lambda) = 10^{10}$ and $\lambda = 4 \times 10^{-7}$ so (2) gives,

$$T = \frac{1.439 \times 10^{-2}}{4 \times 10^{-7} \ln\left[\dfrac{3.742 \times 10^{-16}}{(4 \times 10^{-7}) \times 10^{10}} + 1\right]} = 2380 \text{ K}$$

Interpretation

Since the body is an ideal radiator it will radiate the most possible power per unit area at any given temperature. Consequently any real body would have to be raised to a higher temperature than a black body to obtain the same radiated power per unit area.

Mathematical comment

It is not possible to re-arrange Equation (1) to give λ as a function of $E(\lambda)$ and T. This is due to the way that λ appears twice in the equation i.e. once in a power and once in an exponential. To solve (1) for λ requires numerical techniques but it is possible to use a graphical technique to find the rough value of λ which satisfies (1) for particular values of $E(\lambda)$ and T.

Physics Case Study 2

Black body radiation 2

Mathematical Skills

Topic	Workbook
Logarithms and exponentials	[6]
Numerical solution of equations	[12], [31]

Introduction

A common need in engineering thermodynamics is to determine the radiation emitted by a body heated to a particular temperature at all wavelengths or a particular wavelength such as the wavelength of yellow light, blue light or red light. This would be important in designing a lamp for example. The total power per unit area radiated at temperature T (in K) may be denoted by $E(\lambda)$ where λ is the wavelength of the emitted radiation. It is assumed that a perfect absorber and radiator, called a **black body**, will absorb all radiation falling on it and which emits radiation at various wavelengths λ according to the formula

$$E(\lambda) = \frac{C_1}{\lambda^5\left[e^{C_2/(\lambda T)} - 1\right]} \qquad (1)$$

where $E(\lambda)$ measures the energy (in W m^{-2}) emitted at wavelength λ (in m) at temperature T (in K). The values of the constants C_1 and C_2 are 3.742×10^{-16} W m^{-2} and 1.439×10^{-2} m K respectively. This formula is known as Planck's distribution law. Figure 2.1 shows the radiation $E(\lambda)$ as a function of wavelength λ for various values of the temperature T. Note that both scales are plotted logarithmically. In practice, a body at a particular temperature is not a black body and its emissions will be less intense at a particular wavelength than a black body; the power per unit area radiated by a black body gives the ideal upper limit for the amount of energy emitted at a particular wavelength.

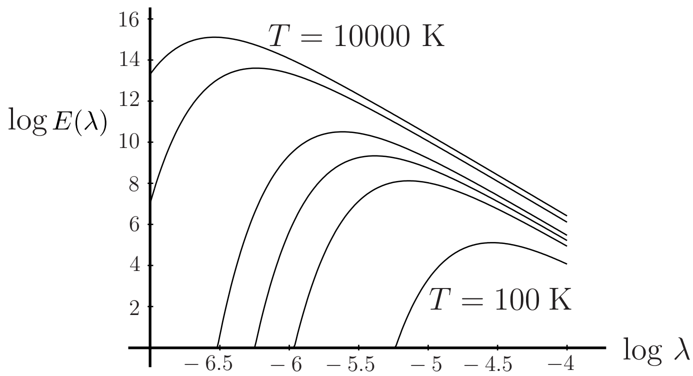

Figure 2.1

The emissive power per unit area $E(\lambda)$ plotted against wavelength (logarithmically) for a black body at temperatures of $T = 100$ K, 400 K, 700 K, 1500 K, 5000 K and 10000 K.

Problem in words

Is it possible to obtain a radiated intensity of 10^8 W m^{-2} at some wavelength for any given temperature?

Mathematical statement of problem

(a) Find possible values of λ when $E(\lambda) = 10^8$ W m^{-2} and $T = 1000$

(b) Find possible values of λ when $E(\lambda) = 10^8$ W m^{-2} and $T = 200$

Mathematical analysis

The graph of Figure 2.2 shows a horizontal line extending at $E(\lambda) = 10^8$ W m^{-2}. This crosses the curve drawn for $T = 1000$K at two points namely once near $\lambda = 10^{-6}$ m and once near $\lambda = 2 \times 10^{-5}$ m. Thus there are two values of λ for which the radiation has intensity $E(\lambda) = 10^8$ W m^{-2} both in the realm of infra-red radiation (although that at $\lambda = 10^{-6}$ m=1 μm is close to the visible light). A more accurate graph will show that the values are close to $\lambda = 9.3 \times 10^{-7}$m and $\lambda = 2.05 \times 10^{-5}$m. It is also possible to use a numerical method such as Newton-Raphson (HELM 12.3 and HELM 31.4) to find these values more accurately. The horizontal line extending at $E(\lambda) = 10^8$ W m^{-2} does not cross the curve for $T = 200$ K. Thus, there is no value of λ for which a body at temperature 200 K emits at $E(\lambda) = 10^8$ W m^{-2}.

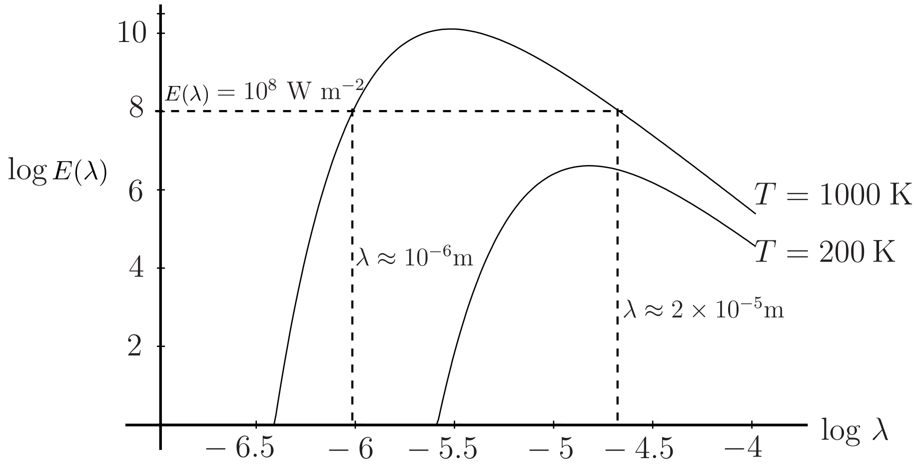

Figure 2.2

The emissive power per unit area $E(\lambda)$ plotted against wavelength (logarithmically) for a black body at temperatures of $T = 200$ K and $T = 1000$ K. For $T = 100$ K an emissive power per unit area of $E(\lambda) = 10^8$ W m^{-2} corresponds to either a wavelength $\lambda \approx 10^{-6}$ or a wavelength $\lambda \approx 2 \times 10^{-5}$. For $T = 200$ K, there is no wavelength λ which gives an emissive power per unit area of $E(\lambda) = 10^8$ W m^{-2}.

Interpretation

Radiation from a black body is dependant both on the temperature and the wavelength. This example shows that it may not be possible for a black body to radiate power at a specific level, irrespective of the wavelength, unless the temperature is high enough.

Physics Case Study 3

Black body radiation 3

Mathematical Skills

Topic	Workbook
Logarithms and exponentials	[6]
Differentiation	[11]

Introduction

A common need in engineering thermodynamics is to determine the radiation emitted by a body heated to a particular temperature at all wavelengths or a particular wavelength such as the wavelength of yellow light, blue light or red light. This would be important in designing a lamp for example. The total power per unit area radiated at temperature T (in K) may be denoted by $E(\lambda)$ where λ is the wavelength of the emitted radiation. It is assumed that a perfect absorber and radiator, called a **black body**, will absorb all radiation falling on it and which emits radiation at various wavelengths λ according to the formula

$$E(\lambda) = \frac{C_1}{\lambda^5 \left[e^{C_2/(\lambda T)} - 1\right]} \qquad (1)$$

where $E(\lambda)$ measures the energy (in W m^{-2}) emitted at wavelength λ (in m) at temperature T (in K). The values of the constants C_1 and C_2 are 3.742×10^{-16} W m^{-2} and 1.439×10^{-2} m K respectively. This formula is known as Planck's distribution law. Figure 3.1 shows the radiation $E(\lambda)$ as a function of wavelength λ for various values of the temperature T. Note that both scales are plotted logarithmically. In practice, a body at a particular temperature is not a black body and its emissions will be less intense at a particular wavelength than a black body; the power per unit area radiated by a black body gives the ideal upper limit for the amount of energy emitted at a particular wavelength.

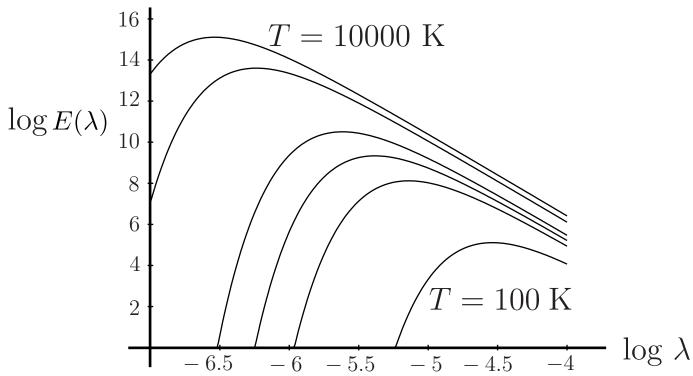

Figure 3.1

The emissive power per unit area $E(\lambda)$ plotted against wavelength (logarithmically) for a black body at temperatures of $T = 100$ K, 400 K, 700 K, 1500 K, 5000 K and 10000 K.

Problem in words

What will be the wavelength at which radiated power per unit area is maximum at any given temperature?

Mathematical statement of problem

For a particular value of T, by means of differentiation, determine the value of λ for which $E(\lambda)$ is a maximum.

Mathematical analysis

Ideally it is desired to maximise

$$E(\lambda) = \frac{C_1}{\lambda^5 \left[e^{C_2/(\lambda T)} - 1\right]}.$$

However, as the numerator is a constant, maximising

$$E(\lambda) = \frac{C_1}{\lambda^5 \left[e^{C_2/(\lambda T)} - 1\right]}$$

is equivalent to minimising the bottom line i.e.

$$\lambda^5 \left[e^{C_2/(\lambda T)} - 1\right].$$

Writing $\lambda^5 \left[e^{C_2/(\lambda T)} - 1\right]$ as y, we see that y can be differentiated by the product rule since we can write

$$y = uv \quad \text{where} \quad u = \lambda^5 \quad \text{and} \quad v = e^{C_2/(\lambda T)} - 1$$

so

$$\frac{du}{d\lambda} = 5\lambda^4$$

and

$$\frac{dv}{d\lambda} = -\frac{C_2}{\lambda^2 T} e^{C_2/(\lambda T)} \quad \text{(by the chain rule), Hence}$$

$$\frac{dy}{d\lambda} = \lambda^5 \left[-\frac{C_2}{\lambda^2 T} e^{C_2/(\lambda T)}\right] + 5\lambda^4 \left[e^{C_2/(\lambda T)} - 1\right]$$

At a maximum/minimum, $\dfrac{dy}{d\lambda} = 0$ hence

$$\lambda^5 \left[-\frac{C_2}{\lambda^2 T} e^{C_2/(\lambda T)}\right] + 5\lambda^4 \left[e^{C_2/(\lambda T)} - 1\right] = 0$$

i.e.

$$-\frac{C_2}{\lambda T} e^{C_2/(\lambda T)} + 5 \left[e^{C_2/(\lambda T)} - 1\right] = 0 \quad \text{(on division by } \lambda^4\text{).}$$

If we write $C_2/(\lambda T)$ as z then $-ze^z + 5\,[e^z - 1] = 0$ i.e.

$$(5 - z)\, e^z = 5 \tag{3}$$

This states that there is a definite value of z for which $E(\lambda)$ is a maximum. As $z = C_2/(\lambda T)$, there is a particular value of λT giving maximum $E(\lambda)$.Thus, the value of λ giving maximum $E(\lambda)$ occurs for a value of T inversely proportional to λ. To find the constant of proportionality, it is necessary to solve Equation (3).

To find a more accurate solution, it is necessary to use a numerical technique, but it can be seen that there is a solution near $z = 5$. For this value of z, e^z is very large ≈ 150 and the left-hand side of (3) can only equal 5 if $5 - z$ is close to zero. On using a numerical technique, it is found that the value of z is close to 4.965 rather than exactly 5.000.

Hence $\quad C_2/\left(\lambda_{\max} T\right) = 4.965 \quad$ so $\quad \lambda_{\max} = \dfrac{C_2}{4.965T} = \dfrac{0.002898}{T}$.

This relationship is called Wein's law:

$$\lambda_{\max} = \frac{C_w}{T}$$

where $C_w = 0.002898$ m K is known as Wein's constant.

Interpretation

At a given temperature the radiated power per unit area from a black body is dependant only on the wavelength of the radiation. The *nature* of black body radiation indicates that there is a specific value of the wavelength at which the radiation is a maximum. As an example the Sun can be approximated by a black body at a temperature of $T = 5800$ K. We use Wein's law to find the wavelength giving maximum radiation. Here, Wein's law can be written

$$\lambda_{\max} = \frac{0.002898}{5800} \approx 5 \times 10^{-7} \text{ m} = 5000 \text{ Å} \quad \text{(to three significant figures)}$$

which corresponds to visible light in the yellow part of the spectrum.

Physics Case Study 4

Black body radiation 4

Mathematical Skills

Topic	Workbook
Logarithms and Exponentials	[6]
Integration	[13]
Numerical Integration	[31]

Introduction

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$$E(\lambda) = \frac{C_1}{\lambda^5\left[e^{C_2/(\lambda T)} - 1\right]} \tag{1}$$

where $E(\lambda)$ measures the energy (in W m^{-2}) emitted at wavelength λ (in m) at temperature T (in K). The values of the constants C_1 and C_2 are 3.742×10^{-16} W m^{-2} and 1.439×10^{-2} m K respectively. This formula is known as Planck's distribution law. Figure 4.1 shows the radiation $E(\lambda)$ as a function of wavelength λ for various values of the temperature T. Note that both scales are plotted logarithmically. In practice, a body at a particular temperature is not a black body and its emissions will be less intense at a particular wavelength than a black body; the power per unit area radiated by a black body gives the ideal upper limit for the amount of energy emitted at a particular wavelength.

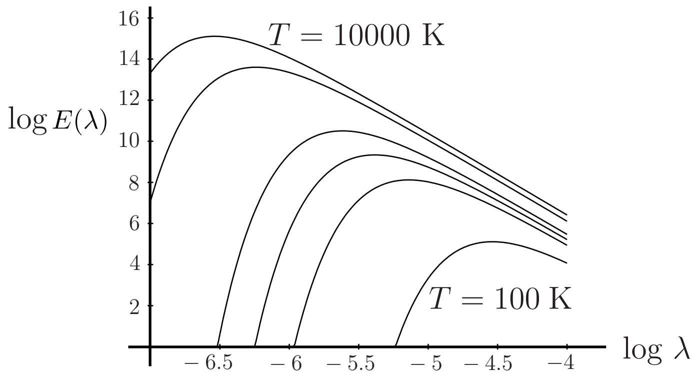

Figure 4.1

The emissive power per unit area $E(\lambda)$ plotted against wavelength (logarithmically) for a black body at temperatures of $T = 100$ K, 400 K, 700 K, 1500 K, 5000 K and 10000 K.

Problem in words

Determine the total power per unit area radiated at all wavelengths by a black body at a given temperature.

The expression (1) gives the amount of radiation at a particular wavelength λ. If this expression is summed across all wavelengths, it will give the total amount of radiation.

Mathematical statement of problem

Calculate

$$E_b = \int_0^\infty E(\lambda)d\lambda = \int_0^\infty \frac{C_1}{\lambda^5\left[e^{C_2/(\lambda T)}-1\right]}d\lambda.$$

Mathematical analysis

The integration can be achieved by means of the substitution $U = C_2/(\lambda T)$ so that

$$\lambda = C_2/(UT), \quad dU = -\frac{C_2}{\lambda^2 T}d\lambda \quad \text{i.e.} \quad d\lambda = -\frac{\lambda^2 T}{C_2}dU.$$

When $\lambda = 0, \quad U = \infty$ and when $\lambda = \infty, \quad U = 0$. So E_b becomes

$$E_b = \int_\infty^0 \frac{C_1}{\lambda^5(e^U-1)}\left(-\frac{\lambda^2 T}{C_2}\right)dU = -\int_\infty^0 \frac{C_1 T}{C_2\lambda^3(e^U-1)}dU$$

$$= \int_0^\infty \frac{C_1 T}{C_2(e^U-1)}\left(\frac{UT}{C_2}\right)^3 dU = T^4\int_0^\infty \frac{C_1}{(C_2)^4}\frac{U^3}{(e^U-1)}dU.$$

The important thing is that E_b is proportional to T^4 i.e. the total emission from a black body scales as T^4. The constant of proportionality can be found from the remainder of the integral i.e.

$$\frac{C_1}{(C_2)^4}\int_0^\infty \frac{U^3}{(e^U-1)}dU \quad \text{where} \quad \int_0^\infty \frac{U^3}{(e^U-1)}dU$$

can be shown by means of the polylog function to equal $\dfrac{\pi^4}{15}$. Thus

$$E_b = \frac{C_1\pi^4}{15(C_2)^4}T^4 = 5.67\times 10^{-8}\ \text{W m}^{-2}\times T^4 = \sigma T^4, \text{ say}$$

i.e.

$$E_b = \sigma T^4$$

This relation is known as the Stefan-Boltzmann law and $\sigma = 5.67\times 10^{-8}$ W m^{-2} is known as the Stefan-Boltzmann constant.

Interpretation

You will no doubt be familiar with Newton's law of cooling which states that bodies cool (under convection) in proportion to the simple difference in temperature between the body and its surroundings. A more realistic study would incorporate the cooling due to the radiation of energy. The analysis we have just carried out shows that heat loss due to radiation will be proportional to the difference in the fourth powers of temperature between the body and its surroundings.

Physics Case Study 5

Amplitude of a monochromatic optical wave passing through a glass plate

Mathematical Skills

Topic	Workbook
Trigonometric functions	[4]
Complex numbers	[10]
Sum of geometric series	[16]

Introduction

The laws of optical reflection and refraction are, respectively, that the angles of incidence and reflection are equal and that the ratio of the sines of the incident and refracted angles is a constant equal to the ratio of sound speeds in the media of interest. This ratio is the index of refraction (n). Consider a monochromatic (i.e. single frequency) light ray with complex amplitude A propagating in air that impinges on a glass plate of index of refraction n (see Figure 5.1). At the glass plate surface, for example at point O, a fraction of the impinging optical wave energy is transmitted through the glass with complex amplitude defined as At where t is the transmission coefficient which is assumed real for the purposes of this Case Study. The remaining fraction is reflected. Because the speed of light in glass is less than the speed of light in air, during transmission at the surface of the glass, it is refracted toward the normal. The transmitted fraction travels to B where fractions of this fraction are reflected and transmitted again. The fraction transmitted back into the air at B emerges as a wave with complex amplitude $A_1 = At^2$. The fraction reflected at B travels through the glass plate to C with complex amplitude rtA where $r \equiv |r|e^{-i\xi}$ is the complex reflection coefficient of the glass/air interface. This reflected fraction travels to D where a fraction of this fraction is transmitted with complex amplitude $A_2 = A\ t^2r^2e^{-i\varphi}$ where φ is the phase lag due to the optical path length difference with ray 1 (see Engineering Example 4 in HELM 4.2). No absorption is assumed here therefore $|t|^2 + |r|^2 = 1$.

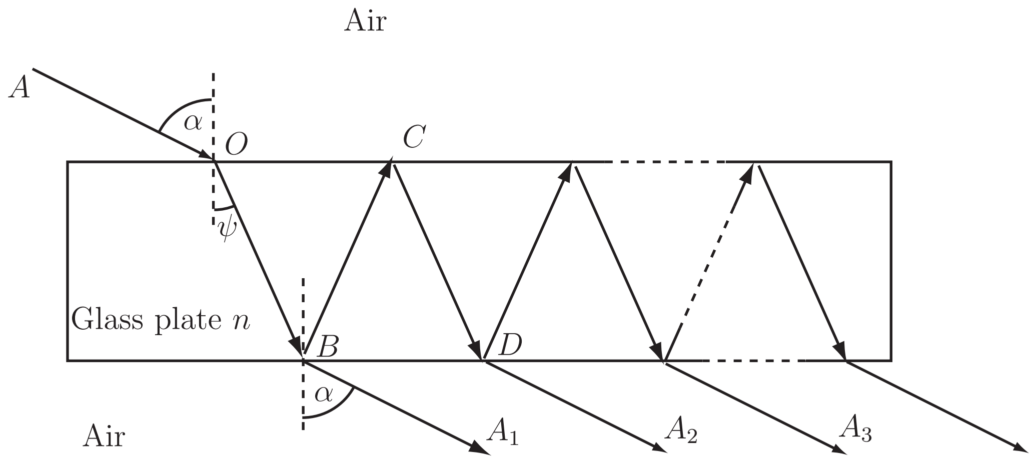

Figure 5.1: Geometry of a light ray transmitted and reflected through a glass plate

Problem in words

Assuming that the internal faces of the glass plate have been treated to improve reflection, and that an infinite number of rays pass through the plate, compute the total amplitude of the optical wave passing through the plate.

Mathematical statement of problem

Compute the total amplitude $\Lambda = \sum_{i=1}^{\infty} A_i$ of the optical wave outgoing from the plate and show that $\Lambda = \dfrac{At^2}{1-|r|^2 e^{-i(\varphi+2\xi)}}$. You may assume that the expression for the sum of a geometrical series of real numbers is applicable to complex numbers.

Mathematical analysis

The objective is to find the infinite sum $\Lambda = \sum_{i=1}^{\infty} A_i$ of the amplitudes from the optical rays passing through the plate. The first two terms of the series A_1 and A_2 are given and the following terms involve additional factor $r^2 e^{-i\varphi}$. Consequently, the series can be expressed in terms of a general term or rank N as

$$\Lambda = \sum_{i=1}^{\infty} A_i = At^2 + At^2 r^2 e^{-i\varphi} + At^2 r^4 e^{-2i\varphi} + \ldots + At^2 r^{2N} e^{-iN\varphi} + \ldots \qquad (1)$$

Note that the optical path length difference creating the phase lag φ between two successive light rays is derived in Engineering Example 4 in HELM 4.2. Taking out the common factor of At^2, the infinite sum in Equation (1) can be rearranged to give

$$\Lambda = At^2[1 + \{r^2e^{-i\varphi}\}^1 + \{r^2e^{-i\varphi}\}^2 + \ldots + \{r^2e^{-i\varphi}\}^N + \ldots]. \qquad (2)$$

The infinite series Equation (2) can be expressed as an infinite geometric series

$$\Lambda = At^2 \lim_{n\to\infty} [1 + q + q^2 + \ldots + q^N + \ldots]. \qquad (3)$$

where $q \equiv r^2 e^{-i\varphi}$. Recalling from HELM 16.1 that for q real $[1 + q + q^2 + \ldots + q^N + \ldots] = \dfrac{1-q^N}{1-q}$ for $q \neq 1$, we will use the extension of this result to complex q. We verify that the condition $q \neq 1$ is met in this case. Starting from the definition of $q \equiv r^2e^{-i\varphi}$ we write $|q| = |r^2e^{-i\varphi}| = |r^2||e^{-i\varphi}| = |r^2|$. Using the definition $r \equiv |r|e^{-i\varphi}, |r^2| = ||r|^2e^{-2i\xi}| = |r|^2|e^{-2i\xi}| = |r|^2$ and therefore $|q| = |r|^2 = 1 - |t|^2$. This is less than 1 because the plate interior surface is not perfectly reflecting. Consequently, $|q| < 1$ i.e. $q \neq 1$. Equation (3) can be expressed as

$$\Lambda = At^2 \lim_{n\to\infty} \left\{ \frac{1-q^N}{1-q} \right\}. \qquad (4)$$

As done for series of real numbers when $|q| < 1,\ \lim_{N\to\infty} q^N = 0$ and Equation (4) becomes

$$\Lambda = At^2 \frac{1}{1 - r^2e^{-i\varphi}}. \qquad (5)$$

Using the definition of the complex reflection coefficient $r \equiv |r|e^{-i\xi}$, Equation (5) gives the final result

$$\Lambda = \frac{At^2}{1-|r|^2e^{-i(\varphi+2\xi)}} \qquad (6)$$

Interpretation

Equation (6) is a complex expression for the amplitude of the transmitted monochromatic light. Although complex quantities are convenient for mathematical modelling of optical (and other) waves, they cannot be measured by instruments or perceived by the human eye. What can be observed is the intensity defined by the square of the modulus of the complex amplitude.

Physics Case Study 6

Intensity of the interference field due to a glass plate

Mathematical Skills

Topic	Workbook
Trigonometric functions	[4]
Complex numbers	[10]

Introduction

A monochromatic light with complex amplitude A propagates in air before impinging on a glass plate (see Figure 6.1). In Physics Case Study 5, the total complex amplitude $\Lambda = \sum_{i=1}^{\infty} A_i$ of the optical wave outgoing from the glass plate was derived as

$$\Lambda = \frac{At^2}{1 - |r|^2 e^{-i(\varphi + 2\xi)}}$$

where t is the complex transmission coefficient and $r \equiv |r|e^{-i\xi}$ is the complex reflection coefficient. ξ is the phase lag due to the internal reflections and φ is the phase lag due to the optical path length difference between two consecutive rays. Note that the intensity of the wave is defined as the square of the modulus of the complex amplitude.

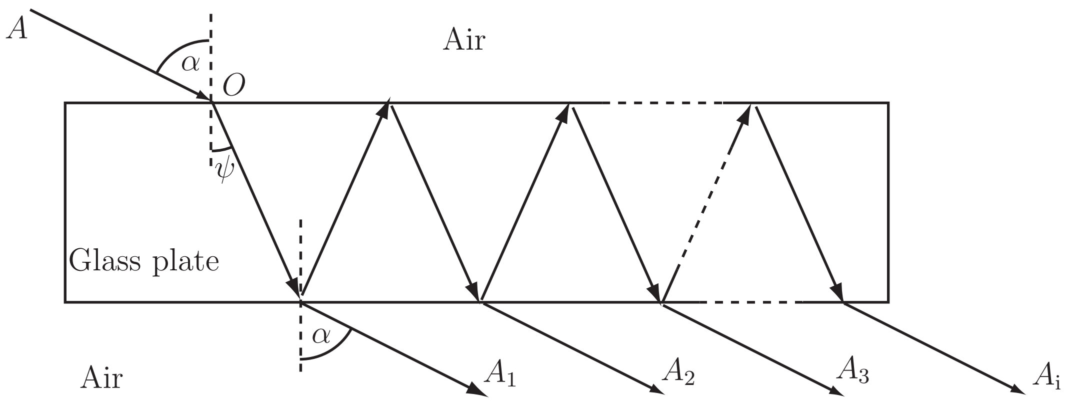

Figure 6.1: Geometry of a light ray transmitted and reflected through a glass plate

Problem in words

Find how the intensity of light passing through a glass plate depends on the phase lags introduced by the plate and the transmission and reflection coefficients of the plate.

Mathematical statement of problem

Defining I as the intensity of the wave, the goal of the exercise is to evaluate the square of the modulus of the complex amplitude expressed as $I \equiv \Lambda\Lambda^*$.

Mathematical analysis

The total amplitude of the optical wave transmitted through the glass plate is given by

$$\Lambda = \frac{At^2}{1 - |r|^2 e^{-i(\varphi+2\xi)}}. \qquad (1)$$

Using the properties of the complex conjugate of products and ratios of complex numbers (HELM 10.1) the conjugate of (1) may be expressed as

$$\Lambda^* = \frac{A^* t^{2^*}}{1 - |r|^2 e^{+i(\varphi+2\xi)}}. \qquad (2)$$

The intensity becomes

$$I \equiv \Lambda\Lambda^* = \frac{AA^* t^2 t^{2^*}}{(1 - |r|^2 e^{-i(\varphi+2\xi)})(1 - |r|^2 e^{+i(\varphi+2\xi)})}. \qquad (3)$$

In HELM 10.1 it is stated that the square of the modulus of a complex number z can be expressed as $|z|^2 = zz^*$. So Equation (3) becomes

$$I = \frac{|A|^2 (tt^*)^2}{(1 - |r|^2 e^{+i(\varphi+2\xi)} - |r|^2 e^{-i(\varphi+2\xi)} + |r|^4)}. \qquad (4)$$

Taking out the common factor in the last two terms of the denominator,

$$I = \frac{|A|^2 |t|^4}{1 + |r|^4 - |r|^2 \{e^{+i(\varphi+2\xi)} + e^{-i(\varphi+2'\xi)}\}}. \qquad (5)$$

Using the exponential form of the cosine function $\cos(\varphi+2\xi) = \{e^{-i(\varphi+2\xi)} + e^{-i(\varphi+2\xi)}\}/2$ as presented in HELM 10.3, Equation (5) leads to the final result

$$I = \frac{|A^2|\ |t|^4}{1 + |r|^4 - 2|r|^2 \cos(2\xi + \varphi)}. \qquad (6)$$

Interpretation

Recall from Engineering Example 4 in HELM 4.3 that φ depends on the angle of incidence and the refractive index of the plate. So the transmitted light intensity depends on angle. The variation of intensity with angle can be detected. A vertical screen placed beyond the glass plate will show a series of interference fringes.

Physics Case Study 7

Propagation time difference between two light rays transmitted through a glass plate

Mathematical Skills

Topic	Workbook
Trigonometric functions	[4]

Introduction

The laws of optical reflection and refraction are, respectively, that the angles of incidence and reflection are equal and that the ratio of the sines of the incident and refracted angles is a constant equal to the ratio of sound speeds in the media of interest. This ratio is the index of refraction (n). Consider a light ray propagating in air that impinges on a glass plate of index of refraction n and of thickness e at an angle α with respect to the normal (see Figure 7.1).

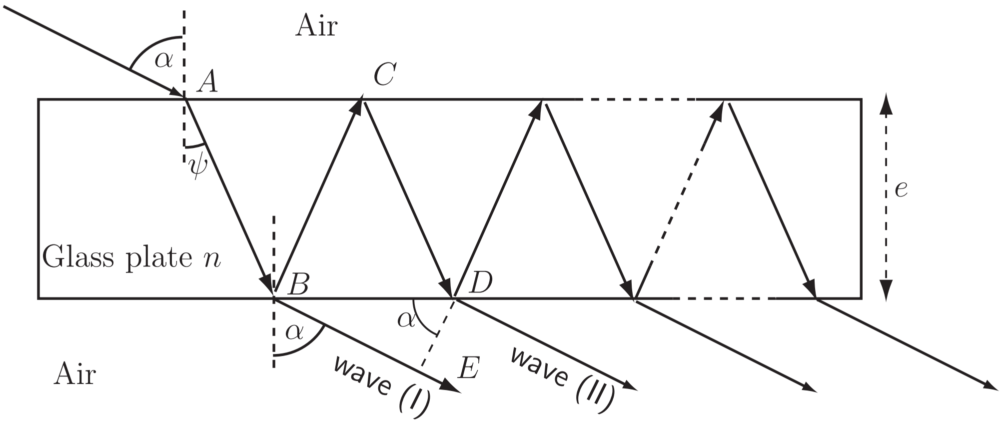

Figure 7.1: Geometry of a light ray transimitted and reflected through a glass plate

At the glass plate surface, for example at point A, a fraction of the impinging optical wave energy is transmitted through the glass and the remaining fraction is reflected. Because the speed of light in glass is less than the speed of light in air, during transmission at the surface of the glass, it is refracted toward the normal at an angle ψ. The transmitted fraction travels to B where a fraction of this fraction is reflected and transmitted again. The fraction transmitted back into the air at B emerges as wave (I). The fraction reflected at B travels through the glass plate to C where a fraction of this fraction is reflected back into the glass. This reflected fraction travels to D where a fraction of this fraction is transmitted as wave (II). Note that while the ray AB is being reflected inside the glass plate at B and C, the fraction transmitted at B will have travelled the distance BE. Beyond the glass plate, waves (I) and (II) interfere depending upon the phase difference between them. The phase difference depends on the propagation time difference.

Problem in words

Using the laws of optical reflection and refraction, determine the difference in propagation times between waves (I) and (II) in terms of the thickness of the plate, the refracted angle, the speed of light in air and the index of refraction. When interpreting your answer, identify three ray paths that are omitted from Figure 1 and state any assumptions that you have made.

Mathematical statement of problem

Using symbols v and c to represent the speed of light in glass and air respectively, find the propagation time difference τ between waves (I) and (II) from $\tau = (BC + CD)/v - BE/c$ in terms of e, n, c and ψ.

Mathematical analysis

The propagation time difference between waves (I) and (II) is given by

$$\tau = (BC + CD)/v - BE/c. \tag{1}$$

As a result of the law of reflection, the angle between the normal to the plate surface and AB is equal to that between the normal and BC. The same is true of the angles to the normal at C, so BC is equal to CD.

In terms of ψ and e

$$BC = CD = e/\cos\psi, \tag{2}$$

so

$$BC + CD = \frac{2e}{\cos\psi}. \tag{3}$$

The law of refraction (a derivation is given in Engineering Example 2 in HELM 12.2), means that the angle between BE and the normal at B is equal to the incident angle and the transmitted rays at B and D are parallel.

So in the right-angled triangle BED

$$\sin\alpha = BE/BD. \tag{4}$$

Note also that, from the two right-angled halves of isosceles triangle ABC,

$$\tan\psi = BD/2e.$$

Replacing BD by $2e\tan\psi$ in (4) gives

$$BE = 2e\tan\psi\sin\alpha. \tag{5}$$

Using the law of refraction again

$$\sin\alpha = n\sin\psi.$$

So it is possible to rewrite (5) as

$$BE = 2en\tan\psi\sin\psi$$

which simplifies to

$$BE = 2en\sin^2\psi/\cos\psi \tag{6}$$

Using Equations (3) and (6) in (1) gives

$$\tau = \frac{2e}{v\cos\psi} - \frac{2en\sin^2\psi}{c\cos\psi}$$

But the index of refraction $n = c/v$ so

$$\tau = \frac{2ne}{c\cos\psi}(1-\sin^2\psi).$$

Recall from HELM 4.3 that $\cos^2\psi \equiv 1-\sin^2\psi$. Hence Equation (6) leads to the final result:

$$\tau = \frac{2ne}{c}\cos\psi \qquad (7)$$

Interpretation

Ray paths missing from Figure 7.1 include reflected rays at A and D and the transmitted ray at C. The analysis has ignored ray paths relected at the 'sides' of the plate. This is reasonable as long as the plate is much wider and longer than its thickness.

The propagation time difference τ means that there is a phase difference between rays (I) and (II) that can be expressed as $\varphi = \dfrac{2\pi c\tau}{\lambda} = \dfrac{4\pi ne\cos\psi}{\lambda}$ where λ is the wave-length of the monochromatic light. The concepts of phase and phase difference are introduced in HELM 4.5 Applications of Trigonometry to Waves. An additional phase shift ξ is due to the reflection of ray (II) at B and C. It can be shown that the optical wave interference pattern due to the glass plate is governed by the phase lag angle $\varphi + 2\xi$. Note that for a fixed incidence angle α (or ψ as the refraction law gives $\sin\alpha = n\sin\psi$), the phase $\varphi + 2\xi$ is constant.

Physics Case Study 8

Fraunhofer diffraction through an infinitely long slit

Mathematical Skills

Topic	Workbook
Trigonometric functions	[4]
Complex numbers	[10]
Maxima and minima	[12]

Introduction

Diffraction occurs in an isotropic and homogeneous medium when light does not propagate in a straight line. This is the case, for example, when light waves encounter holes or obstacles of size comparable to the optical wavelength. When the optical waves may be considered as plane, which is reasonable at sufficient distances from the source or diffracting object, the phenomenon is known as Fraunhofer diffraction. Such diffraction affects all optical images. Even the best optical instruments never give an image identical to the object. Light rays emitted from the source diffract when passing through an instrument aperture and before reaching the image plane. Fraunhofer diffraction theory predicts that the complex amplitude of a monochromatic light in the image plane is given by the Fourier transform of the aperture transmission function.

Problem in words

Express the far-field intensity of a monochromatic light diffracted through an infinitely long slit-aperture characterised by a uniform transmission function across its width. Give your result in terms of the slit-width and deduce the resulting interference fringe pattern. Deduce the changes in the fringe system as the slit-width is varied.

Mathematical statement of problem

Suppose that $f(x)$ represents the transmission function of the slit aperture where the variable x indicates the spatial dependence of transmission through the aperture on the axis perpendicular to the direction of the infinite dimension of the slit. A one-dimensional function is sufficient as it is assumed that there is no variation along the axis of the infinitely long slit. Fraunhofer Diffraction Theory predicts that the complex optical wave amplitude $F(u)$ in the image plane is given by the Fourier transform of $f(x)$ i.e. $F(u) = F\{f(x)\}$. Since the diffracted light intensity $I(u)$ is given by the square of the modulus of $F(u)$, i.e. $I(u) = |F(u)|^2 = F(u)F(u)^*$, the fringe pattern is obtained by studying the minima and maxima of $I(u)$.

Mathematical analysis

Represent the slit width by $2a$. The complex amplitude $F(u)$ can be obtained as a Fourier transform $F(u) = F\{f(x)\}$ of the transmission function $f(x)$ defined as

$$f(x) = 1 \text{ for } -a \leq x \leq a, \tag{1a}$$

$$f(x) = 0 \text{ for } -\infty < x < -a \text{ and } a < x < \infty. \tag{1b}$$

with $f(x) = 1$ or $f(x) = 0$ indicating maximum and minimum transmission respectively, corresponding to a completely transparent or opaque aperture. The required Fourier transform is that of a rectangular pulse (see Key Point 2 in HELM 24.1). Consequently, the result

$$F(u) = 2a\frac{\sin ua}{ua} \tag{2}$$

can be used. The sinc function, $\sin(ua)/ua$ in (2), is plotted in HELM 24.1 page 8 and reproduced below as Figure 8.1.

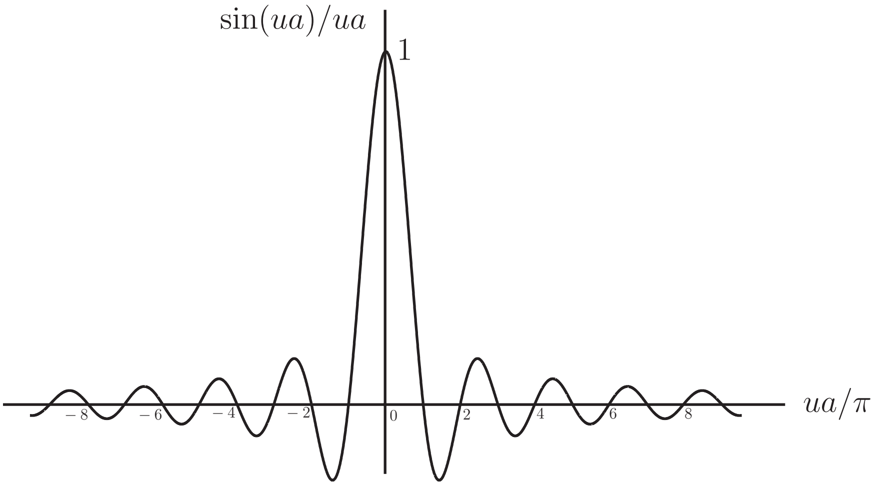

Figure 8.1: Plot of sinc function

$F(u)$ has a maximum value of $2a$ when $u = 0$. Either side of the maximum, Figure 8.1 shows that the sinc curve crosses the horizontal axis at $ua = n\pi$ or $u = n\pi/a$, where n is a positive or negative integer. As u increases, $F(u)$ oscillates about the horizontal axis. Subsequent stationary points, at $ua = (2n+1)\pi/2,\ (|u| \geq \pi/a)$ have successively decreasing amplitudes. Points $ua = 5\pi/2, 9\pi/2 \ldots$ etc., are known as secondary maxima of $F(u)$.

The intensity $I(u)$ is obtained by taking the product of (2) with its complex conjugate. Since $F(u)$ is real, this is equivalent to squaring (2). The definition $I(u) = F(u)F(u)^*$ leads to

$$I(u) = 4a^2\left(\frac{\sin ua}{ua}\right)^2. \tag{3}$$

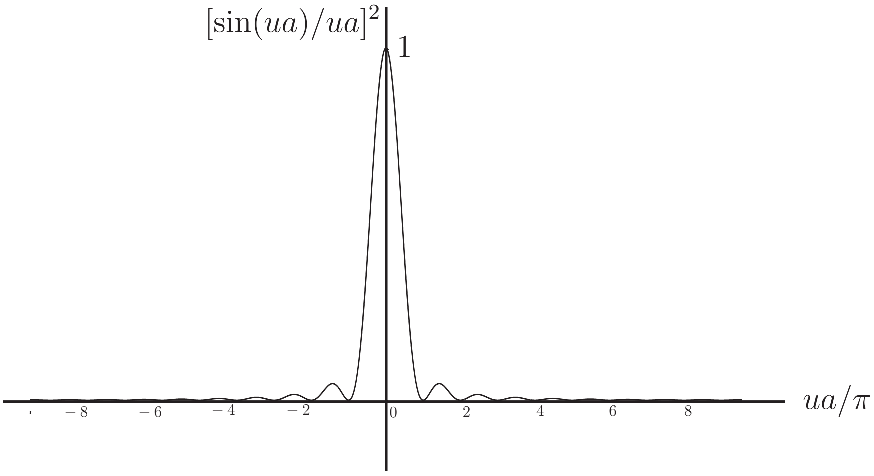

Figure 8.2: Plot of square of sinc function

$I(u)$ differs from the square of the sinc function only by the factor $4a^2$. For a given slit width this is a constant. Figure 8.2, which is a plot of the square of the sinc function, shows that the intensity $I(u)$ is always positive and has a maximum value $I_{\text{max}} = 4a^2$ when $u = 0$. The first intensity minima either side of $u = 0$ occur for $ua = \pm\pi$. Note that the secondary maxima have much smaller amplitudes than that of the central peak.

Interpretation

The transmission function $f(x)$ of the slit aperture depends on the single spatial variable x measured on an axis perpendicular to the direction of the infinite dimension of the slit and no variation of the intensity is predicted along the projection of the axis of the infinitely long slit on the image plane. Consequently, the fringes are parallel straight lines aligned with the projection of the axis of infinite slit length on the image plane (see Figure 8.3). The central fringe at $u = 0$ is bright with a maximum intensity $I_{\text{max}} = 4a^2$ while the next fringe at $u = \pm\pi/a$ is dark with the intensity approaching zero. The subsequent bright fringes (secondary maxima) are much less bright than the central fringe and their brightness decreases with distance from the central fringe.

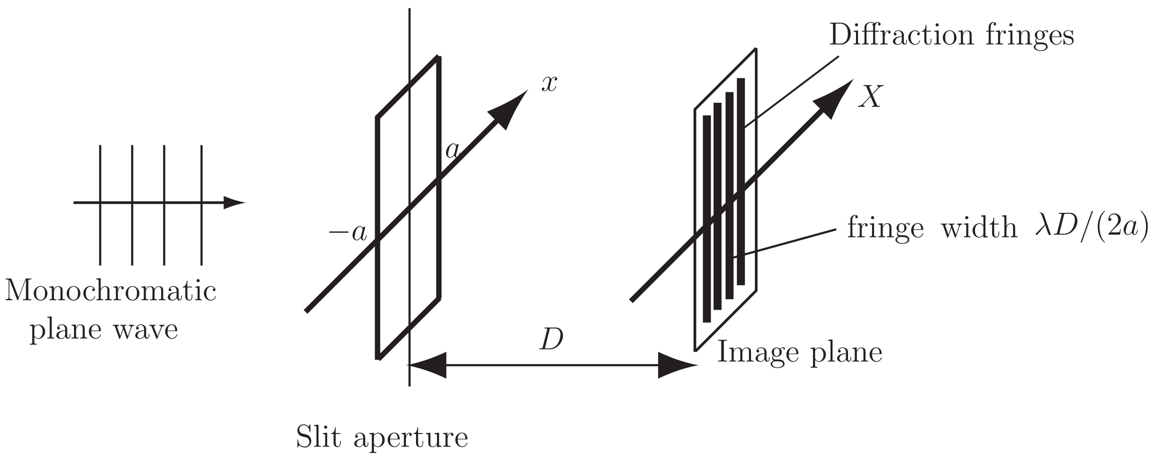

Figure 8.3: Geometry of monochromatic light diffraction through an idealised infinite slit aperture

As the slit-width is increased or decreased, the intensity of the bright central fringe respectively increases or decreases as the square of the slit-width. The Fourier transform variable u is assumed to be proportional to the fringe position X in the image plane. Therefore, as the slit-width a is increased or decreased, the fringe spacing π/a decreases or increases accordingly. It can be shown from diffraction theory that

$$u = \frac{2\pi X}{\lambda D}$$

where λ is the wavelength, D is the distance between the image and aperture planes, and X is the position in the image plane. When $ua = \pm\pi$, $\dfrac{2\pi Xa}{\lambda D} = \pm\pi$ so the first dark fringe positions are given by $X = \pm\,\dfrac{\lambda D}{2a}$. This means that longer wavelengths and longer aperture/image distances will produce wider bright fringes.

Physics Case Study 9

Fraunhofer diffraction through an array of parallel infinitely long slits

Mathematical Skills

Topic	Workbook
Trigonometric functions	[4]
Exponential function	[6]
Complex numbers	[10]
Maxima and minima	[12]
Sum of geometric series	[16]
Fourier transform of a rectangular pulse	[24]
Shift and linearity properties of Fourier transforms	[24]

Introduction

Diffraction occurs in an isotropic and homogeneous medium when light does not propagate in a straight line. This is the case, for example, when light waves encounter holes or obstacles of size comparable to the optical wavelength. When the optical waves may be considered as plane, which is reasonable at sufficient distances from the source or diffracting object, the phenomenon is known as Fraunhofer diffraction. Such diffraction affects all optical images. Even the best optical instruments never give an image identical to the object. Light rays emitted from the source diffract when passing through an instrument aperture and before reaching the image plane. Fraunhofer diffraction theory predicts that at sufficient distance from the diffracting object the complex amplitude of a monochromatic light in the image plane is given by the Fourier transform of the aperture transmission function.

Problem in words

(i) Deduce the light intensity due to a monochromatic light diffracted through an aperture consisting of a single infinitely long slit, characterised by a uniform transmission function across its width, when the slit is shifted in the direction of the slit width.
(ii) Calculate the light intensity resulting from transmission through N parallel periodically spaced infinitely long slits.

Mathematical statement of problem

(i) Suppose that $f(x-l)$ represents the transmission function of the slit aperture where the variable x indicates the spatial dependence of the aperture's transparency on an axis perpendicular to the direction of the infinite dimension of the slit and l is the distance by which the slit is shifted in the negative x-direction. A one-dimensional function is appropriate as it is assumed that there is no variation in the transmission along the length of the slit. The complex optical wave amplitude $G(u)$ in the image plane is give by the Fourier transform of $f(x-l)$ denoted by $G(u) = F\{f(x-l)\}$. The intensity of the diffracted light $I_1(u)$ is given by the square of the modulus of $G(u)$

$$I_1(u) = |F\{f(x-l)\}|^2 = |G(u)|^2. \qquad (1)$$

(ii) In the image plane, the total complex amplitude of the optical wave generated by N parallel identical infinitely long slits with centre-to-centre spacing l is obtained by summing the amplitudes diffracted by each aperture. The resulting light intensity can be expressed as the square of the modulus of the Fourier transform of the sum of the amplitudes. This is represented mathematically as

$$I_N(u) = |F\{\sum_{n=1}^{N} f(x - nl)\}|^2. \tag{2}$$

Mathematical analysis

(i) Result of shifting the slit in the direction of the slit width

Assume that the slit width is 2a. The complex optical amplitude in the image plane $G(u)$ can be obtained as a Fourier transform $G(u) = F\{f(x - l)\}$ of the transmission function $f(x - l)$ defined as

$$f(x - l) = 1 \text{ for } -a - l \leq x \leq a - l, \tag{3a}$$

$$f(x - l) = 0 \text{ for } -\infty < x < -a - l \text{ and } a - l < x < \infty. \tag{3b}$$

The maximum and minimum transmission correspond to $f(x-l) = 1$ and $f(x-l) = 0$ respectively. Note that the function $f(x - l)$ centred at $x = l$ defined by (3a)-(3b) is identical to the function $f(x)$ centred at the origin but shifted by l in the negative x-direction.

The shift property of the Fourier transform introduced in subsection 2 of HELM 24.2 gives the result

$$F\{f(x - l)\} = e^{-iul} F\{f(x - l)\} = e^{-iul} G(u). \tag{4}$$

Combining Equations (1) and (4) gives

$$I_1(u) = |e^{-iul} G(u)|^2. \tag{5}$$

The complex exponential can be expressed in terms of trigonometric functions, so

$$|e^{-iul}|^2 = |\cos(ul) - i\sin(ul)|^2.$$

For any complex variable, $|z|^2 = zz^*$, so

$$|e^{-iul}|^2 = [\cos(ul) - i\sin(ul)][\cos(ul) + i\sin(ul)]$$

$$= \cos^2(ul) - i^2\sin^2(ul).$$

Since $i^2 = -1$,

$$|e^{-iul}|^2 = \cos^2(ul) + \sin^2(ul) = 1.$$

The Fourier transform $G(u)$ is that of a rectangular pulse, as stated in Key Point 2 in subsection 3 of HELM 24.1, so

$$G(u) = 2a\frac{\sin ua}{ua} \tag{6}$$

Consequently, the light intensity

$$I_1(u) = 4a^2\left(\frac{\sin ua}{ua}\right)^2. \tag{7}$$

This is the same result as that obtained for diffraction by a slit centered at $x = 0$.

Interpretation

No matter where the slit is placed in the plane parallel to the image plane, the same fringe system is obtained.

(ii) Series of infinite slits

Consider an array of N parallel infinitely long slits arranged periodically with centre-to-centre spacing l. The resulting intensity is given by Equation (2). The linearity property of the Fourier transform (see subsection 1 of HELM 24.2) means that

$$F\{\sum_{n=1}^{N} f(x-nl)\} = \sum_{n=1}^{N} F\{f(x-nl)\} \tag{8}$$

Using Equation (4) in (8) leads to

$$F\{\sum_{n=1}^{N} f(x-nl)\} = \sum_{n=1}^{N} e^{-iunl}G(u). \tag{9}$$

The function $G(u)$ is independent of the index n, therefore it can be taken out of the sum to give

$$F\{\sum_{n=1}^{N} f(x-nl)\} = G(u)\sum_{n=1}^{N} e^{-iunl}. \tag{10}$$

Taking the common factor e^{-iul} out of the sum leads to

$$\sum_{n=l}^{N} e^{-iunl} = e^{-iul}\{1 + e^{-iul} + e^{-iu2l} + \ldots e^{-iu(N-1)l}\}. \tag{11}$$

The term in brackets in (11) is a geometric series whose sum is well known (see HELM 16.1).

Assuming that the summation formula applies to complex numbers

$$\sum_{n=l}^{N} e^{-iunl} = e^{-iul}\frac{1-[e^{-iul}]^N}{1-e^{-iul}}. \tag{12}$$

Using (12) and (10) in (2) gives an expression for the light intensity

$$I_N(u) = \left|G(u)e^{-iul}\frac{1-e^{-iuNl}}{1-e^{-iul}}\right|^2. \tag{13}$$

Recalling that the modulus of a product is the same as the product of the moduli, (13) becomes

$$I_N(u) = |G(u)|^2\left|e^{-iul}\right|^2\left|\frac{1-e^{-iuNl}}{1-e^{-iul}}\right|^2. \tag{14}$$

Using $|e^{-iul}|^2 = 1$ in (14) leads to

$$I_N(u) = I_1(u)\left|\frac{1-e^{-iuNl}}{1-e^{-iul}}\right|^2. \tag{15}$$

The modulus of the ratio of exponentials can be expressed as a product of the ratio and its conjugate which gives

$$\left|\frac{1-e^{-iuNl}}{1-e^{-iul}}\right|^2 = \frac{(1-e^{-iuNl})(1-e^{iuNl})}{(1-e^{-iul})(1-e^{iul})} = \frac{(2-e^{iuNl}-e^{-iuNl})}{2-e^{iul}-e^{-iul}}.$$

Using the definition of cosine in terms of exponentials (see HELM 10.3),

$$\left|\frac{1-e^{-iuNl}}{1-e^{-iul}}\right|^2 = \frac{1-\cos(uNl)}{1-\cos(ul)}.$$

Using the identity $1-\cos(2\theta) \equiv 2\sin^2\theta$ gives

$$\left|\frac{1-e^{-iuNl}}{1-e^{-iul}}\right|^2 = \frac{\sin^2\left(\dfrac{uNl}{2}\right)}{\sin^2\left(\dfrac{ul}{2}\right)}. \qquad (16)$$

Using (16) and (5) in (15) leads to the final result for the intensity of the monochromatic light diffracted through a series of N parallel infinitely long periodically spaced slits:

$$I_N(u) = 4a^2\left(\frac{\sin ua}{ua}\right)^2\left(\frac{\sin\left(\dfrac{uNl}{2}\right)}{\sin\left(\dfrac{ul}{2}\right)}\right)^2. \qquad (17)$$

Interpretation

The transmission function $f(x)$ of a single slit depends on the single spatial variable x measured on an axis perpendicular to the direction of the infinite dimension of the slit. The linearity and shift properties of the Fourier transform show that a one-dimensional intensity function of diffracted light is obtained with N identical periodic slits. Consequently, no variation of the intensity is predicted along the projection of the axis of infinite slit length on the image plane. Therefore, the diffraction interference fringes are straight lines parallel to the projection of the axis of the infinite slit on the image plane.

In the expression for the light intensity after diffraction through the N slits, the first term

$$4a^2\left(\frac{\sin ua}{ua}\right)^2$$

is the function corresponding to the intensity due to one slit.

The second factor

$$\left(\frac{\sin\left(\dfrac{Nul}{2}\right)}{\sin\left(\dfrac{ul}{2}\right)}\right)^2$$

represents the result of interference between the waves diffracted through the N slits.

Physics Case Study 10 studies the graphical form of a normalised version of the function in (17) for the case of two slits $(N = 2)$. It is found that the oscillations in intensity, due to the interference term, are bounded by an envelope proportional to the intensity due to one slit.

Physics Case Study 10

Interference fringes due to two parallel infinitely long slits

Mathematical Skills

Topic	Workbook
Trigonometric functions	[4]
Complex numbers	[10]
Maxima and minima	[12]
Maclaurin series expansions	[16]

Introduction

Diffraction occurs in an isotropic and homogeneous medium when light does not propagate in a straight line. This is the case for example, when light waves encounter holes or obstacles of size comparable to the optical wavelength. When the optical waves may be considered as plane, which is reasonable at sufficient distances from the source or diffracting object, the phenomenon is known as **Fraunhofer diffraction**. Such diffraction affects all optical images. Even the best optical instruments never give an image identical to the object. Light rays emitted from the source diffract when passing through an instrument aperture and before reaching the image plane. Prediction of the intensity of monochromatic light diffracted through N parallel periodically spaced slits, idealised as infinite in one direction, is tackled in Physics Case Study 9. The resulting expression for intensity divided by a^2, $2a$ being the slit width, is

$$J_N(u) = 4\left(\frac{\sin ua}{ua}\right)^2 \left(\frac{\sin\left(\frac{uNl}{2}\right)}{\sin\left(\frac{ul}{2}\right)}\right)^2 \qquad (1)$$

where l is the centre-to-centre spacing of the slits and u represents position on an axis perpendicular to that of the infinite length of the slits in the image plane. The first term is called the **sinc** function and corresponds to the intensity due to a single slit (see Physics Case Study 8). The second term represents the result of interference between the N slits.

Problem in words

On the same axes, plot the components (sinc function and interference function) and the normalised intensity along the projection of the slit-width axis on the image plane for a monochromatic light diffracted through two 2 mm wide infinite slits with 4 mm centre-to-centre spacing. Describe the influence of the second component (the interference component) on the intensity function.

Mathematical statement of problem

Plot $y = 4\left(\dfrac{\sin ua}{ua}\right)^2$, $y = \left(\dfrac{\sin(ul)}{\sin\left(\dfrac{ul}{2}\right)}\right)^2$ and $y = J_2(u) = 4\left(\dfrac{\sin ua}{ua}\right)^2 \times \left(\dfrac{\sin(ul)}{\sin\left(\dfrac{ul}{2}\right)}\right)^2$ on the same graph for $a = 1$ mm, $l = 4$ mm and $N = 2$.

[Note that as $\sin(ul) \equiv 2\sin\left(\dfrac{ul}{2}\right)\cos\left(\dfrac{ul}{2}\right)$ the expression $\dfrac{\sin(ul)}{\sin\left(\dfrac{ul}{2}\right)}$ simplifies to $2\cos\left(\dfrac{ul}{2}\right)$.]

Mathematical analysis

The dashed line in Figure 10.1 is a plot of the function

$$4\left(\frac{\sin ua}{ua}\right)^2 \qquad (3)$$

The horizontal axis in Figure 10.1 is expressed in units of $\pi/l,\ (l = 4a)$, since this enables easier identification of the maxima and minima. The function in (3) involves the square of a ratio of a sine function divided its argument. It has minima (which have zero value, due to the square) when the numerator $\sin(ua) = 0$ and when the denominator $ua \neq 0$. If n is a positive or negative integer, these conditions can be written as $u = n\pi/a$ and $u \neq 0$ respectively. Alternatively, since $l = 4a$, the conditions can be written as $ul/\pi = 4n\ (n \neq 0)$. This determines the minima of (3) (see Figure 10.1). The first minima are at $n = \pm 1$, i.e. $ul/\pi = \pm 4$. The dashed line shows a maximum at $ul/\pi = 0$ i.e. $u = 0$. When both the sine function and its argument tend to zero $(u = 0)$, the first term in the Maclaurin series expansion of sine (see HELM 16.5) gives the ratio $[(ua)/(ua)]^2 = 1$. So the maximum of (3) at $ul/\pi = 0$ has the value 4. Note that the subsequent maxima of the function (3) are at $ul/\pi = \pm 6$ and the function is not periodic.

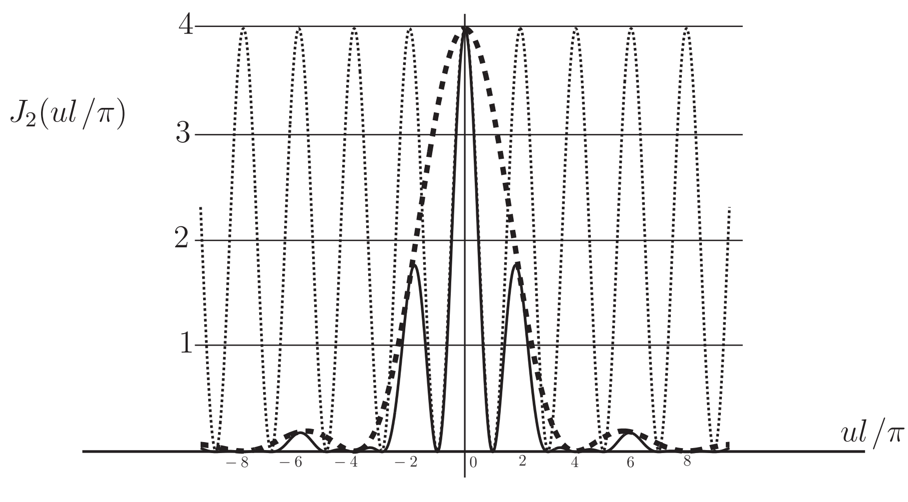

Figure 10.1: Plots of the normalised intensity $J_2(ul/\pi)$ and its component functions

The dotted line is a graph of the interference term

$$\left(\frac{\sin(ul)}{\sin\left(\dfrac{ul}{2}\right)}\right)^2 \quad \text{which is equivalent to} \quad 4\cos^2\left(\frac{ul}{2}\right) \qquad (4)$$

The interference function (4) is the square of the ratio of two sine functions and has minima (zeros because of the square) when the numerator is $\sin(ul) = 0$ and when the denominator is $\sin(ul/2) \neq 0$. If n is a positive or negative integer, these two conditions can be written as $ul/\pi = n$ and $ul/\pi \neq 2n$. Both conditions are satisfied by the single condition $ul/\pi = (2n+1)$. As a result of scaling the horizontal axis in units of π/l, the zeros of (4) coincide with positive or negative odd integer values on this axis.

When the sine functions in the numerator and denominator of (4) tend to zero simultaneously, the first terms in their Maclaurin series expansions give the ratio $[(ul)/(ul/2)]^2 = 4$ which means that the maxima of (4) have the value 4. They occur when $ul/\pi = 2n$ (see Figure 10.1). Note that the function obtained from the square of the ratio of two functions with periods 2 and 4 (in units of π/l) is a function of period 2.

The solid line is a plot of the product of the functions in (3) and (4) (i.e. Equation (2)) for a slit semi-width $a = 1$ mm and a slit separation with centre-to-centre spacing $l = 4$ mm. For convenience in plotting, the product has been scaled by a factor of 4. Note that the oscillations of the solid line are like those of function (4) but are bound by the dashed line corresponding to the squared sinc function (3). Function (3) is said to provide the **envelope** of the product function (2). The solid line shows a principal maximum at $u = 0$ and secondary maxima around $|u/\pi| \approx 2$ with about half the intensity of the principal maximum. Subsequent higher order maxima show even lower magnitudes not exceeding $1/10^{\text{th}}$ of the principal maximum.

Interpretation

The diffraction interference fringes are parallel straight lines aligned with the projection of the axis of infinite slit length on the image plane as seen in Figure 10.2. The central fringe at $u = 0$ is bright with a maximum normalised intensity $J_2(0) = 4$. Either side of the central fringe at $u/\pi = 1,\ (ul/\pi = 4)$, is dark with the intensity approaching zero. The next bright fringes have roughly half the brightness of the central fringe and are known as secondary. Subsequent bright fringes show even lower brightness. Note that the fringe at $ul/\pi = 6$ is brighter than those at $ul/\pi \approx 3.5$ and $ul/\pi \approx 4.5$ due to the envelope function (3).

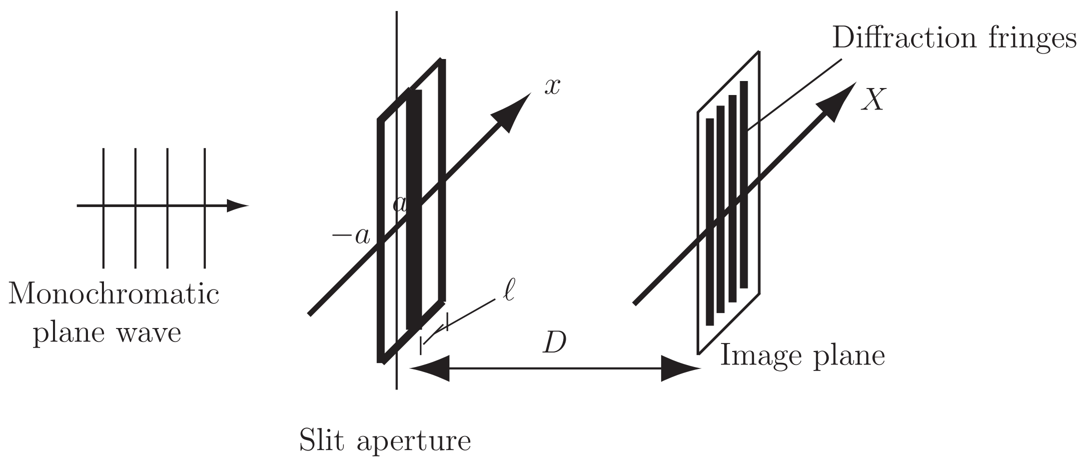

Figure 10.2: Monochromatic light diffraction through two slits

Physics Case Study 11

Acceleration in polar coordinates

Mathematical Skills

Topic	Workbook
Vectors	[9]
Polar coordinates	[17]

Introduction

Consider the general planar motion of a point P whose position is given in polar coordinates. The point P may represent, for example, the centre of mass of a satellite in the gravitational field of a planet.

The position of a point P can be defined by the Cartesian coordinates (x, y) of the position vector $\underline{OP} = x\underline{i} + y\underline{j}$ as shown in Figure 11.1.

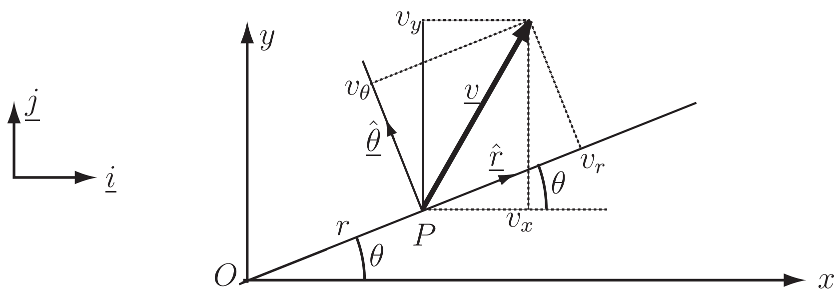

Figure 11.1: Cartesian and polar coordinate system

If $\underline{i}$ and $\underline{j}$ denote the unit vectors along the coordinate axes, by expressing the basis vector set $(\underline{i}, \underline{j})$ in terms of the set $(\hat{\underline{r}}, \hat{\underline{\theta}})$ using trigonometric relations, the components (v_r, v_θ) of velocity $\underline{v} = v_r\hat{\underline{r}} + v_\theta\hat{\underline{\theta}}$ can be related to the components (v_x, v_y) of $\underline{v} = v_x\underline{i} + v_y\underline{j}$ expressed in terms of $(\underline{i}, \underline{j})$.

Problem in words

Express the radial and angular components of the velocity and acceleration in polar coordinates.

Mathematical statement of problem

The Cartesian coordinates can be expressed in terms of the polar coordinates (r, θ) as

$$x = r\cos\theta \qquad (1)$$

and

$$y = r\sin\theta. \qquad (2)$$

The components (v_x, v_y) of velocity $\underline{v} = v_x\underline{i} + v_y\underline{j}$ in the frame $(\underline{i}, \underline{j})$ are derived from the time derivatives of (1) and (2). $\dfrac{d\underline{i}}{dt} = \underline{0}$ and $\dfrac{d\underline{j}}{dt} = \underline{0}$ as the unit vectors along Ox and Oy are fixed with time. The components (a_x, a_y) of acceleration in the frame $(\underline{i}, \underline{j})$ are obtained from the time derivative of $\underline{v} = \frac{d}{dt}(\underline{OP})$, and use of expressions for (v_r, v_θ) leads to the components (a_r, a_θ) of acceleration.

Mathematical analysis

The components (v_x, v_y) of velocity $\underline{v}$ are derived from the time derivative of (1) and (2) as

$$v_x = \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta \tag{3}$$

$$v_y = \frac{dy}{dt} = \frac{dr}{dt}\sin\theta - r\frac{d\theta}{dt}\cos\theta. \tag{4}$$

The components (a_x, a_y) of acceleration are obtained from the time derivatives of (3) and (4),

$$a_x = \frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta - r\left(\frac{d\theta}{dt}\right)^2\cos\theta - r\frac{d^2\theta}{dt^2}\sin\theta \tag{5}$$

$$a_y = \frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - r\left(\frac{d\theta}{dt}\right)^2\sin\theta + r\frac{d^2\theta}{dt^2}\cos\theta. \tag{6}$$

The components (v_x, v_y) of velocity $\underline{v} = v_x\underline{i} + v_y\underline{j}$ and (a_x, a_y) of acceleration $\underline{a} = a_x\underline{i} + a_y\underline{j}$ are expressed in terms of the polar coordinates. Since the velocity vector $\underline{v}$ is the same in both basis sets,

$$v_x\underline{i} + v_y\underline{j} = v_r\underline{\hat{r}} + v_\theta\underline{\hat{\theta}}. \tag{7}$$

Use of known expresions for the basis vector $(\underline{i}, \underline{j})$ in terms of the basis vectors $(\underline{\hat{r}}, \underline{\hat{\theta}})$ leads to expressions for (v_r, v_θ) in terms of the polar coordinates.

Projection of the basis vectors $(\underline{i}, \underline{j})$ onto the basis vectors $(\underline{\hat{r}}, \underline{\hat{\theta}})$ leads to (see Figure 11.2)

$$\underline{i} = \cos\theta\underline{\hat{r}} - \sin\theta\underline{\hat{\theta}} \tag{8}$$

$$\underline{j} = \sin\theta\underline{\hat{r}} + \cos\theta\underline{\hat{\theta}} \tag{9}$$

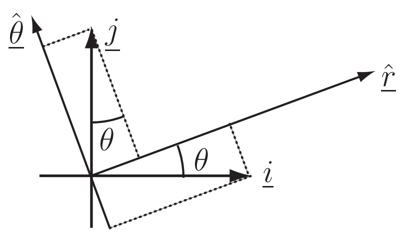

Figure 11.2: Projections of the Cartesian basis set onto the polar basis set

Equation (7) together with (8) and (9) give

$$v_x(\cos\theta\underline{\hat{r}} - \sin\theta\underline{\hat{\theta}}) + v_y(\sin\theta\underline{\hat{r}} + \cos\theta\underline{\hat{\theta}}) = v_r\underline{\hat{r}} + v_\theta\underline{\hat{\theta}}. \quad (10)$$

The basis vectors $(\underline{\hat{r}}, \underline{\hat{\theta}})$ have been chosen to be independent, therefore (10) leads to the two equations,

$$v_r = v_x\cos\theta + v_y\sin\theta \quad (11)$$

$$v_\theta = -v_x\sin\theta + v_y\cos\theta. \quad (12)$$

Using (3) and (4) in (11) and (12) gives

$$v_r = \frac{dr}{dt} \quad (13)$$

$$v_\theta = r\frac{d\theta}{dt} \quad (14)$$

Following the same method for the components (a_r, a_θ) of the acceleration,
the equation $\underline{a} = a_x\underline{i} + a_y\underline{j} = a_r\underline{\hat{r}} + a_\theta\underline{\hat{\theta}}$ allows us to write

$$a_r = a_x\cos\theta + a_y\sin\theta \quad (15)$$

$$a_\theta = -a_x\sin\theta + a_y\cos\theta. \quad (16)$$

Using (5) and (6) in (15) and (16) leads to

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad (17)$$

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}. \quad (18)$$

Interpretation

The angular velocity $\dfrac{d\theta}{dt}$ and acceleration $\dfrac{d^2\theta}{dt^2}$ are often denoted by ω and α respectively. The component of velocity along $\underline{\hat{\theta}}$ is $r\omega$. The component of acceleration along $\underline{\hat{r}}$ includes not only the so-called radial acceleration $\dfrac{d^2r}{dt^2}$ but also $-r\omega^2$ the centripetal acceleration or the acceleration toward the origin which is the only radial terms that is left in cases of circular motion. The acceleration along $\underline{\hat{\theta}}$ includes not only the term $r\alpha$, but also the Coriolis acceleration $2\dfrac{dr}{dt}\omega$. These relations are useful when applying Newton's laws in a polar coordinate system. Engineering Case Study 13 in HELM 48 uses this result to derive the differential equation of the motion of a satellite in the gravitational field of a planet.